PhD Macro Notes

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Acknowledgements/Structure

This is centered around my notes and notes from Craig Burnside's Macro I course, which is largely based on the Stokey&Lucas and Ljungqvist&Sargent texts. My notes from Greg Givens' Macro I course were also invaluable and used to help provide more context, intuition, and guidance on structuring and notation.

My formatting for these notes may be a bit too cumbersome for some readers; with the exception of the Dynamic Programming section, there is generally limited spacing/headings. This was developed as a reference for myself, so the linear structuring and compact delivery of content was what I felt like would best keep me in the "flow of logic/learning/retention" after trial and error in several Macro courses. Also, it should be noted that the sections are meant to be read chronologically, as some of the content/intuition is nested.

Also, I will keep an index/list/dictionary of abbreviations that I did not define explicitly.

1 Solow

Properties: We don't specify preferences (instead use 'ad hoc' specifications), discrete time ($t = 1, 2, \dots$) with one infinitely-existing, single-member household and one consumption good

Economy: Production is given by $y_t = F(k_t, h_t)$. We assume that this function is CRTS and increasing in both inputs with diminishing marginal returns. Since both inputs are positive, we consider their marginal products to be infinite at 0 and 0 at infinity (in the limit).

Constraints: We have a resource constraint and an evolution of capital equation

$$c_t + i_t = y_t \ \underline{\textbf{and}}\ k_{t+1} = (1-\delta)k_t + i_t$$

'Ad Hoc' Investment: Normalize by $h_t = 1$, so $y_t = f(k_t)$. We assume that no matter what the economy produces, we will invest a fixed share, in other words $i_t = sf(k_t)$. This implies that $c_t = (1-s)f(k_t)$.

Steady State (s.s): Let k denote steady state value ($k_t = k_{t+1}$). From evolution of capital equation, the steady state yields the implied condition that $sf(k) = \delta k$. This means capital is depreciating at the exact rate its being invested in. This gives strong implications for the evolution of capital.

Transition Dynamics: Consider a graph of k_{t+1} vs k_t, where we plot the capital evolution equation as well as $k_{t+1} = k_t$ (45° line). Given the 45° line is a formalization of the steady state definition, it intersects the evolution of capital equation at the s.s. If the economy is investing more than the value of depreciated capital, then $k_{t+1} > k_t$. If the opposite is true, the inequality sign is flipped. This relates directly to capital's relationship with the s.s. Given $k_0 > 0$, **capital converges monotonically towards the steady state**.

Proof: Consider the case where $k_0 < k$

$$g(k_t) = sf(k_t) + (1-\delta)k_t \implies g'(k_t) = sf'(k_t) + (1-\delta) > 0$$

Note by Euler's theorem and CRTS $f(k_t) = F(k_t, 1) = 1 \cdot F_h(k_t, 1) + k_t F_k(k_t, 1) > k_t f'(k_t)$ since $F_h(k_t, 1) > 0$.

$$\implies 1 > \delta = \frac{sf(k_t)}{k_t} > sf'(k_t) \implies k_t f'(k_t) - f(k_t) < 0 \ \underline{\textbf{and}}\ g'(k_t) = sf'(k_t) + 1 - \delta < 1$$

So $g'(\cdot)$ is strictly positive and bounded above. Thus by fundamental theorem of calculus

$$k_1 - k = g(k_0) - g(k) = -\int_{k_0}^{k} g'(x)dx < -\int_{k_0}^{k} 1dx < 0$$

$$\textbf{also note } \frac{d}{dk_t}\frac{f(k_t)}{k_t} = k_t^2(k_t f'(k_t) - f(k_t)) < 0 \implies \frac{k_1 - k_0}{k_0} = \frac{sf(k_0)}{k_0} - \delta > \frac{sf(k)}{k} - \delta = 0$$

since $k_0 < k$ and we have a decreasing function. So $k_1 \in (k_0, k)$ and by induction $k_{t+1} \in (k_t, k)$ ■.
This idea is furthered by considering a quantification of the intertemporal growth rate of capital:
$\gamma = \frac{k_{k+1} - k_t}{k_t} = \frac{sf(k_t)}{k_t} - \delta \implies \frac{\partial \gamma}{\partial k_t} = \frac{s}{k_t}(k_t f'(k_t) - f(k_t)) = \frac{-s}{k_t}F_h < 0$ (by Euler's theorem).
This implies that capital accumulation slows the larger a capital stock is

Golden Rule: The golden rule level is what savings rate will yield the largest steady state of consumption. While preferences aren't specified, we assume that on balance having more to consume means more utility. So we can find the golden rule rate by defining steady state consumption and other terms as function (of s)

$$c(s) = f(k(s)) - \delta k(s) \implies \frac{\partial c}{\partial s} = (f_k(k(s)) - \delta)\frac{\partial k}{\partial s}$$

We make this into a F.O.C since we want to maximize. Focusing on the differenced term

$$0 = (f_k(k(s)) - \delta)\frac{\partial k}{\partial s} \implies k^{GR} = f_k^{-1}(\delta) \implies s^{GR} = \frac{\delta k^{GR}}{f(k^{GR})}$$

Consider the locus of points, $f(k_t) - \delta k_t$, derived from imposing the steady state in the equation for c_t, then reintroducing the time subscripts. Then $c_t = (1 - s^{\mathrm{GR}})f(k_t)$ intersects this locus at $(k^{\mathrm{GR}}, c^{\mathrm{GR}})$, its apex.

2 Small, Open Econ Neoclassical

2.1 w/ Exogenous Income

Properties: In this version, there is no capital and income (y_t) is treated as known for all time periods and given/exogenous, as in the actions of the population do not affect the resources available to consume (could think of it like fruit falling from a tree). Households can have some control over their intertemporal income by borrowing in a "zero coupon" bond market, where the rest of the world (ROW) can be counted on to supply demand. The sequence of bond prices $\{q_t\}$ is also exogenous and known. We also assume that in this model that households have "time preferences", in that they care more about present day consumption than consuming in the future. To formalize this, we consider $\beta \in (0,1)$ as the preference, with $\beta = \frac{1}{1+\rho}$ and the closer β is to zero, the more the HH prefers consuming today over future periods.

Constraints: Notationally, let b_{t+1} be the number of bonds purchased at time t. This is because we consider them paying off one unit of consumption good at time $t+1$, but purchased at a cost of q_t (per bond) at time t. More explicitly, the resources gained from bonds bought at time t is the per-bond payoff (which is 1, in consumption-good units) multiplied by the number of bonds purchased (b_{t+1}), yielding a payoff of $1 \cdot b_{t+1}$ in time $t+1$. The cost is similarly derived as the per-bond price (q_t) multiplied by the number of bonds purchased at time t (b_{t+1}), so the cost incurred from bond buying at time t is $q_t \cdot b_{t+1}$. So in period t, the HH has $b_t + y_t$ as income (the exogenous income stream plus the payoff from the bonds it purchased in the previous period) and the costs incurred are what it consumes plus the bonds it is purchasing in period t ($c_t + q_t \cdot b_{t+1}$). Thinking of bonds as a saving mechanism, it should be obvious that resources gained in period t must be equivalent to resources spent. Thus, $c_t + q_t b_{t+1} = y_t + b_t$. To further emphasize bonds as a means of saving, we can think of the rate of return on bonds (interest rate) as the size of the difference between bond payout and bond price relative to price, giving $r_t = \frac{1-q_t}{q_t} = q_t^{-1} - 1$. Then define $\widetilde{b}_{t+1} \equiv q_t b_{t+1}$, which yields a more precise constraint of

$$c_t + \widetilde{b}_{t+1} = y_t + (1 + r_{t-1})\widetilde{b}_t$$

Note that $\widetilde{b}_t$ represents net foreign assets, $-(\widetilde{b}_{t+1} - \widetilde{b}_t)$ is the capital account balance, and $r_{t-1}\widetilde{b}_t = (1-q_{t-1})b_t$ is income from the ROW. So the budget constraint gives a nice visualization of the current account balance $(y_t + r_{t-1}\widetilde{b}_t - c_t)$ being equal to the capital account balance. We also need to consider something called the "no-ponzi" condition, formed the idea that the household will want to borrow infinitely if left unrestrained. Obviously, this is infeasible from a practical perspective because lenders would incur huge debts that would never be repaid. So to induce feasibility in the bond-market, the no-ponzi condition we will consider here is $\lim_{t\to\infty} q^t b_t \geq 0$, which the household will select as 0 to expend all of its resources.

Economy: We assume lifetime utility is $\sum_{t=0}^{\infty} \beta^t u(c_t)$. We assume that the utility function is increasing with diminishing marginal returns and an infinite marginal product when consumption approaches 0. The household takes the initial endowment of bonds (b_0) as given and makes choices for the sequence of $\{c_t, b_{t+1}\}$ in order to maximize lifetime utility. This is formalized in the usual Lagrangian paradigm by

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t (y_t + b_t - c_t - q_t b_{t+1})$$

Take F.O.C's for the choice variables and the Lagrange multiplier. Scrolling forward the F.O.C for c_t then substituting into the F.O.C for bonds yields the Euler equation

$$q_t = \beta \frac{u_c(c_{t+1})}{u_c(c_t)} \implies \frac{1+\rho}{1+r_t} = \frac{u_c(c_{t+1})}{u_c(c_t)}$$

$\therefore$ intertemporal marginal rate of substitution $\equiv$ relative price, in time t units, of consuming one unit at $t+1$

Transition Dynamics: We can derive some intuition from the second optimally condition. If $r_t = \rho$, $u_c(c_t) = u_c(c_{t+1}) \implies c_t = c_{t+1}$. If $r_t > \rho$, $u_c(c_t) > u_c(c_{t+1}) \implies c_t < c_{t+1}$, or qualitatively the household has a stronger incentive to save. If $r_t < \rho$, then quantitatively and practically the opposite holds. Notice that the path/changes consumption takes are not affected by income level. But deeper investigation reveals that the present value of current and future income does affect the level of consumption. To see this, consider the special case where $r_t = r = \rho \implies q = \beta$ and $c_t = c$. Then the period 0 B.C is $c_0 + qb_1 = y_0 + b_0 \implies c_0 + qc_1 + q^2 b_2 = y_0 + qy_1 + b_0 \implies \cdots \implies \sum_{t=0}^{\infty} q^t c_t = \sum_{t=0}^{\infty} q^t y_t + b_0$ by substituting for bonds and the non-ponzi condition. Imposing our special case we have $c = (1-q)\sum_{t=0}^{\infty} q^t y_t + b_0$. So essentially, the agent just "annuitizes" (divies up equally) the present value of lifetime resources. There is not a nice closed form solution for non-constant interest rates, but the intuition is the same. We can also get some insight into the nature of intertemporal consumption substitution by performing a first order Taylor expansion (we will assume equality) about the steady state and $r_t = \rho$

$$u_c(c)+u_{cc}(c_t-c) = u_c(c)+u_{cc}(c)(c_{t+1}-c)+\frac{1}{1+\rho}u_c(c)(r_t-\rho) \implies \frac{u_{cc}(c)}{u_c(c)}(c_t-c) = \frac{u_{cc}(c)}{u_c(c)}(c_{t+1}-c)+\frac{1}{1+\rho}(r_t-p)$$

Let $\widehat{x}_t = \frac{x_t - x}{x}$ and $\sigma = \frac{-u_{cc}(c)\cdot c}{u_c(c)}$ denote the *coefficient of relative risk aversion.* Then

$$\frac{u_{cc}(c)\cdot c}{u_c(c)}\widehat{c}_t = \frac{u_{cc}(c)\cdot c}{u_c(c)}\widehat{c}_{t+1} + \frac{\rho}{1+\rho}\widehat{r}_t \implies \widehat{c}_{t+1} - \widehat{c}_t = \sigma^{-1}\cdot\frac{\rho}{1+\rho}\widehat{r}_t$$

When the nominal interest rate is above ρ, the household is incentivized to save and the LHS (and obviously RHS) is greater than zero. But what's the mechanism for this? σ is essentially a scaling factor; the larger it is the less they are willing to intertermporally substitute.

2.2 w/ Production

Properties/Updated Constraints: Now capital is included. We still assume no leisure and normalize by $h_t = 1$ and have a bond market in place. Thus, the new BC is $c_t + k_{t+1} - (1-\delta)k_t + q_t b_{t+1} = f(k_t) + b_t$.

Economy: Now the HH has an additional choice variable: tomorrow's capital. Otherwise the problem is the same with an updated budget constraint:

$$\mathcal{L} = \sum_{t=0}^{\infty}\beta^t u(c_t) + \sum_{t=0}^{\infty}\lambda_t(f(k_t) + (1-\delta)k_t - k_{t+1} - c_t - q_t b_{t+1} + b_t)$$

The FOCs yield the same optimality condition of $q_t = \beta\frac{u_c(c_{t+1})}{u_c(c_t)}$. The FOC with respect to k_{t+1} for the lifetime constraint gives the rate of return to capital between t and $t+1$. So also get the condition that $q_t^{-1} - 1 = r_t = f_k(k_{t+1}) - \delta$ (return on bonds equals return on capital), and if interest rates are constant the economy jumps straight into a steady state. This implies that local preferences don't affect the optimal level of capital. This is because capital is guided by an implicit "no-arbitrage" condition: it's return can't differ from the return on bonds at equilibrium. If it was at a larger steady state capital level, HH would sell capital to finance bonds. At a lesser steady state, it would borrow to acquire more capital since at the level capital has a higher return than it costs to borrow. With respect to a special case that $r_t = r = \rho$, define the corresponding level of capital by $k^{\text{NM}} = f_k^{-1}(\rho+\delta)$. Since the marginal product of capital is higher at lower values of k (and $\rho > 0$), we know that $k^{\text{NM}} < k^{\text{GR}}$. We can also work through this differently/more explicitly: similarly to the exogenous income model, this special case yields constant bond prices and the budget constraint can be written as an infinite sum as in 2.1 ($\sum_{t=0}^{\infty} q^t(c_t + i_t) = \sum_{t=0}^{\infty} q^t y_t + b_0$). Given that $c_t + i_t = c + \delta k$ for $t > 0$ because c_t, k_{t+1} are constant $\forall t$, this leads to

$$\frac{1}{1-q}(c + q\delta k - qf(k)) = f(k_0) + b_0 + (1-\delta)k_0 - k$$

Solving for c and taking a FOC yields k^{NM}, showing that the golden rule is not the optimal choice. However, if you do not take the initial level of capital as given, you would get the golden rule.

Transition Dynamics Consider again $r_t = r = \rho$. Also impose that $k_0 = k^{\text{NM}}$. Thus $y_t = y^{\text{NM}} = f(k^{\text{NM}})$ and $i_t = i^{\text{NM}} = (\delta - 1)k^{\text{NM}} + k^{\text{NM}} = \delta k^{\text{NM}}$. Reusing the above summation process for the budget constraint and taking limits to infinity, the no-ponzi condition implies $c = c^{\text{NM}} = f(k^{\text{NM}}) - \delta k^{\text{NM}}$. To assess intertemporal affect, suppose instead $k_0 < k^{\text{NM}}$. So the economy jumps to a steady state at $t = 1$. So $f(k_0) < y^{\text{NM}}$ and $i_0 = k^{\text{NM}} + (1-\delta)k_0 > k^{\text{NM}} > i^{\text{NM}}$. Recall that we impose that $c_t = c$ (even at period 0). Therefore, the budget constraint summation process from earlier yields

$$c = \frac{q}{1-q}(y^{\text{NM}} - c - i^{\text{NM}}) + y_0 - c_0 = qc^{\text{NM}} + (1-q)(y_0 - i_0) = c^{\text{NM}} + (1-q)[(y_0 - y^{\text{NM}}) - (i_0 - i^{\text{NM}})] < c^{\text{NM}}$$

So because the HH had less income in period 0, it had to invest more to jump to k^{NM}, leaving less lifetime resources to annuitize consumption. The same jump happens if $k_0 > k^{\text{NM}}$. One may raise a question of why a HH would not try to reach the golden rule level of consumption by jumping to the golden rule capital. This is again because of the implicit no-arbitrage condition. Recall that the return to capital is $f_k(k_{t+1}) - \delta$. Since $f_k(k^{\text{GR}}) = \delta$, the return to capital in this case is 0, which is less than our case of $r = \rho$. The consequence of this is that if $k_0 \geq k^{\text{GR}}$, the HH can attain $c > c^{\text{NM}}$

Proof: Assume $k_0 = k^{\text{GR}}$ and $b_0 = 0$. Then consumption and capital are at a steady state for $t \geq 1$. So referencing the BC summation analysis from above we see that

$$\frac{1}{1-q}(c + q(\delta k - f(k))) = (1-\delta)k^{\text{GR}} - k + f(k^{\text{GR}})$$

Think of this as the first equation. Now separately, impose that k and c are at the golden rule

$$\frac{1}{1-q}(c^{\text{GR}} + q(\delta k^{\text{GR}} - f(k^{\text{GR}}))) = (1-\delta)k^{\text{GR}} - k^{\text{GR}} + f(k^{\text{GR}})$$

Now multiplying by $1-q$ and taking the difference between the first and second equations

$$c - c^{\text{GR}} = qf(k) - q\delta k - (1-q)k - qf(k^{\text{GR}}) + q\delta k^{\text{GR}} + (1-q)k^{\text{GR}}$$

The RHS has diminishing marginal returns in k (concave), so we maximize the difference by solving the FOC $qf_k(k) = q\delta + 1 - q$. This yeilds $f_k^{-1}(\delta + q^{-1} - 1)$, in other words k^{NM}. Since $c - c^{\text{GR}}$ is obviously 0 at the golden rule, we must have $c^{\text{NM}} - c^{\text{GR}} > 0$.

3 Dynamic Programming

Prelude/Intuition

Our goal with dynamic programming is to be able to represent an infinite problem compactly (using time-invariant notation). We will build to this structure by formulating the social planning problem using an approach which emphasizes the number periods between relevant "events", then arrive at the point where we can remove the subscripts using some math and new concepts (e.g. Bellman equations and envelope conditions). Again assume the usual properties for resources/utility/production: CRTS, normalizing by $h_t = 1$,limit conditions, increasing with diminishing returns, and $c_t + k_{t+1} - (1-\delta)k_t = f(k)$. Now also consider the function $g(k) \equiv f(k)+(1-\delta)k$ for simplicity. Note that the case where $g(k_t) = k_{t+1}$ is the instance where all resources are allocated to investment, so this is a ceiling on the next periods capital (with 0 being a floor). Solving for consumption, the social planning problem becomes $\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u[g(k_t) - k_{t+1}]$, subject to the aforementioned bound on capital, positive consumption, and an initial endowment of capital. The FOC for tomorrows capital yields the Euler equation with slightly modified notation: $\frac{u_c(g(k_t)-k_{t+1})}{u_c(g(k_{t+1})-k_{t+2})} = \beta g_k(k_{t+1})$. Now consider a period $t = T$. We can define the social planner's decision at any time period using notation with respect to T. If we consider T to be the "final period", then obviously we will want to have used all the economy's resources by the end, meaning $k_{T+1} = 0$. If we begin to start from the end (from this condition) and work backwards, this is known as a **recursive** solution. Now, define the object that we are maximizing in each period as the **objective function**, and the objective function at the optimum as the **value function**. Define the value function at n periods away from $t = T$ by $v_n(\cdot)$. Similarly, define the function which returns optimal choice of capital n periods away from T by $h_n(\cdot)$, known as the **policy function**. So per previous discussion, $h_0(k_T) = k_{T+1} = 0$. Thus, the value function at the terminal date is

$$v_0(k_T) = \max_{k_{T+1}} u[g(k_T) - h_0(k_T)] = u[g(k_T)]$$

Now consider the planner's decision at $T-1$, which is

$$\max_{k_T,k_{T+1}} u[g(k_{T-1}) - k_T] + \beta u[g(k_T) - k_{T+1}] \equiv \max_{k_T}\{u[g(k_{T-1}) - k_T] + \beta \max_{k_{T+1}} u[g(k_T) - k_{T+1}]\}$$

subject to the aforementioned constraints on capital. Note this is $v_1(k_{T-1})$. Also note that $v_0(k_T)$ appears in the RHS of the expression. Therefore this gives us

$$v_1(k_{T-1}) = \max_{k_T} \; u[g(k_{T-1}) - k_T] + \beta v_0(k_T)$$

Consider doing this for $v_2(k_{T-2})$; the same substitution opportunity exists. Therefore, we can generalize this approach by

$$v_s(k_{T-s}) = \max_{k_{T-s+1}} u[g(k_{T-s}) - k_{T-s+1}] + \beta v_{s-1}(k_{T-s+1})$$

with a sequence of optimal capital stocks generated by $k_1 = h_T(k_0), \ldots, k_{T+1} = 0 = h_0(k_T) = h_0(h_1(k_{T-1}))$

Recursive, Time-Invariant Structure

Now we have arrived at an ability to describe the optimal choice of capital irregardless of the period t. Therefore, we can remove the time subscripts from notation and place the emphasis on the given level of capital, considering that we will know what capital we have at the beginning of each period. This is useful for transitioning back into an environment with no terminal date. Let x be the realization of x "today", x' be the realization tomorrow, x'' the realization the day after, and so on. Thus, we conjecture that there is a "timeless optimization", and this gives us the following **Bellman equation**:

$$v(k) = \max_{k'} u[g(k) - k'] + \beta v(k')$$

General Form:

$$v(x) = \max_{x' \in \Gamma(x)} F(x, x') + \beta v(x') \tag{1}$$

subject to $0 \leq k' \leq g(k)$ and $k' = h(k)$, for our specific case. However, we still haven't formally proved that the value function we described exists. Moreover, we haven't proven the existence and uniqueness of the policy function. As stated earlier, the policy function for capital provides a mapping of today's capital to the optimal choice for tomorrow. We conjecture that the policy function is continuous, single-valued, strictly increasing, will always be less(more) than the steady state if the input is less(more) than the steady state, returns the steady state at the steady state, and limits to 0 as $k \to 0$ from the right. This also allows for the formulation of a single-valued and continuous policy function for present day consumption, given by $c(k) = g(k) - h(k)$, which also has the same limit property. However, we need to establish that all this actually holds. Specifically, one may wonder how can we guarantee convergence of the function; that it holds as an "infinite"/timeless principle when all we know is the initial information. Our previous analysis was based on the fact that we ended at "T", but that is not how high-level problems are designed. We will do this by formulating parameters for a general dynamic programming problem and then applying it.

General Dynamic Optimization

Now we are thinking in a purely mathematical sense, not even thinking of an "economy" or even what the variables in our system mean. Consider the problem $\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} F(x_t, x_{t+1})$ such that $x_{t+1} \in \Gamma(x_t)$, with x_0 given. This gives us an equivalent dynamic programming problem of $v(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y)$. In order to conform to some of the paradigms of necessary theorems, we need to define an *operator*. We consider a function, for example, as a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$. An operation maps a function to a function, in other words a function in some space to another, possibly the same, space (e.g. $T : \mathcal{F} \to \mathcal{F}$). Our operator of choice is defined by, given some function w, $w = T(w)$. We also notationally say $w = Tw$ when the functional equation reaches a "fixed" or optimal point; you could consider this a quasi-steady state for the function when it can't be "refined" any further. Then we can write the Bellman equation as a **functional equation**, something that is defined over a function instead of points in a space, such as

$$(Tw)(x) \equiv \max_{y \in \Gamma(x)} F(x, y) + \beta w(y) \tag{2}$$

Where w for the Bellman we are interested in eventually being this fixed point and our value function. But we still have more work to do to leave conjecture land.

Assumptions

We need to make further assumptions to arrive at our goals, which we will denote **(A#)**. Define a feasible sequence of plans. given x_0, by $\Pi(x_0) = \{\{x_t\} : x_{t+1} \in \Gamma(x_t) \ \ \forall\, t > 0\}$. Let $A = \{(x, y) \in X \times X | y \in \Gamma(x)\}$.

(A1) - $\Gamma(x)$ is nonempty $\forall x \in X$
(A2) - X is a convex subset of $\mathbb{R}^l$, and $\Gamma : X \to X$ is nonempty, compact-valued, and continuous.
(A3) - $F : A \to \mathbb{R}$ is bounded and continuous
(A4) - $\forall x_0 \in X$ and $\mathbf{x} \in \Pi(x_0), u(\mathbf{x}) \equiv \lim_{n\to\infty} \sum_{t=\infty}^{n} \beta^t F(x_t, x_{t+1})$,exists (could be infinite limit).
(A5) - F is strictly concave.
(A6) - (convexity) $x, x' \in X, y \in \Gamma(x), y' \in \Gamma(x')$, and $\lambda \in [0,1] \implies \lambda y + (1-\lambda)y' \in \Gamma[\lambda x(1-\lambda)x']$
(A7) - (monotonicity) $x \leq x' \implies \Gamma(x) \subseteq \Gamma(x')$.
(A8) - For each y, $F_x(x, y) > 0$
(A9) - F is continuously differentiable on $\text{Int}(A)$.

If we want all results we will show simultaneously, we can simply assume they all hold, but we will specify later what results require which assumptions. Before we go delving into the results, let's consider some context about what these assumptions mean in the context of our problem and why they're necessary. Note that the $\Gamma(\cdot)$ correspondence is going to be $[0, g(k)]$. The Heine-Borel tells us a set in the reals is compact iff it's closed and bounded. So this immediately should make the assumptions we made for the correspondence seem extremely reasonable. Further, remembering that $F(\cdot)$ is analogous to the utility function, those functional assumptions should also seem extremely non-restrictive, given ones that have already been and will be asserted about the utility function throughout this document and in Macro PhD courses in general.

Notationally, this can get a bit messy because some assumptions and results are stated much cleaner with time subscripts. So just keep any mind x and y are proxies for x_t and x_{t+1} in our case, but they are more general objects related both being in a set X and by a "policy" correspondence where the realization of x affects the possible values y can take on. Another complication is the "general" vs "specific" forms of the value and policy functions. There will be a $\cdot^*$ to refer to the non-recursive formulations, discussed at length in the next section, and also some notation to separate the general case vs. our case. For example, $V(x)$ for a general functional equation and $v(x)$, from (1), when thinking of what we want specifically to correspond to the Bellman equation and using H for the most general case, π when we apply it to the value function we want, and h when we talk about capital.

Equivalence to Sequence Problem

The traditional, non-recursive, solution in the social planning paradigm can be thought of as a *sequence problem* (SP), where we pick an infinite sequence of values of capital, given an initial point. The functional equation (FE) solution says that we can instead just decompose the problem into two parts: maximization today and the maximization problem we will face tomorrow. Obviously if you go down the "rabbit hole", you will see that tomorrow's maximization problem includes the day after, and so on. So it should make sense that these two formulations are equivalent. To see this mathematically, consider the following **example**. Let $V^*(k_0) = \sum_{t=0}^{\infty} \beta^t U(k_t^*, k_{t+1}^*)$ be the sequence problem solution from the given k_0, where k_t^* is the optimal value of capital at date t that we would observe in the SP. Note how the subscript on the input for the solution matches the beginning date of the summation. Then,

$$V^*(k_0) = U(k_0, k_1^*) + \beta U(k_1^*, k_2^*) + \cdots = U(k_0, k_1^*) + \sum_{t=1}^{\infty} \beta^t U(k_t^*, k_{t+1}^*) = U(k_0, k_1^*) + \beta \sum_{t=1}^{\infty} \beta^{t-1} U(k_t^*, k_{t+1}^*) = U(k_0, k_1^*) + \beta V^*(k_1)$$

So it should follow how our operator T will be at a fixed point for our correct Bellman equation. Our policy function can then be defined as the value of tomorrow's capital that satisfies this Bellman equation, and is thus corresponds to the $t+1$ element of the SP solution. Some, including Dr. Burnside, may criticize the structuring, seen in Stokey&Lucas and here, of discussing the SP equivalence to the recursive formulation because we haven't formally established the existence of the value function. However, when we reference the SP example above, it should be obvious that the value function exists and we should have some intuitive understanding of what it is. Further, in my view, the issue of equivalence takes precedent over the issue of uniqueness. Finally, recall the terms SP and FE and note the equation numbering. We will state some "theorems" that will be delineated by their result. Keep in mind the general definition: $V^*(x_0) = \sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*)$

(Equivalence of Values) Given A1-A4, then for any $x \in X$, any solution to the SP ($V^*(x)$) is also a solution to the to the FE, and any solution $v(x)$, from (1), to the FE is a solution to the SP, so that $V^*(x) = v(x) \ \forall x \in X$.

Recall $\Pi(x_0)$, a feasible sequence of plans. Say $\mathbf{x}^* \in \Pi(x_0)$ *attains* $V^*(x_0)$ in the SP if it achieves an equal value of (discounted) lifetime utility.
(Attaining Optimality) Given A1-A4 and $\mathbf{x}^* \in \Pi(x_0)$ that attains $V^*(x_0)$ in the SP, then for $x_t^* \in \mathbf{x}^*$, $V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*)$. Also, a $\mathbf{x}^* \in \Pi(x_0)$ satisfying $V^*(x_t^*)$ attains $V^*(x_0)$ in the SP.

Another way to tackle this: let $v^*(x_0) \equiv \sup_{\mathbf{x} \in \Pi(x_0)} \lim_{n \to \infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ be a modified version of the SP. Given A1, A4, and $\lim_{n \to \infty} \beta^n v(x_n) = 0 \ \forall \mathbf{x} \in \Pi(x_0)$, v^* is a solution to $\sup_{y \in \Gamma(x)} F(x,y) + \beta v(y)$. Further, given A1 and A4 and the supposition that a solution exists, any plan that is optimal (attains the sup) can be generated from the the policy correspondence $H^*(x) = \{y \in \Gamma(x) | v^*(x) = F(x,y) + \beta v^*(y)\}$. Yet will still don't know if the plan to attain this supremum solution is feasible. So note that given a feasible plan $\mathbf{x}^* \in \Pi(x_0)$ that satisfies the SP for all t and that $\lim_{t \to \infty} \beta^t v(x_t^*) = 0$, then $\mathbf{x}^*$ attains the supremum. As mentioned, we need some more math to pin down the uniqueness of the solution and the value function.

Math Background

Now we will define some more mathematical principles to get to some important notation/results. A *metric space* is a set $\mathcal{S}$ and a function (*metric*) $\rho : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ such that $\rho(f,g) \geq 0 \ \forall f,g \in \mathcal{S}$ (equality holds iff $f = g$), $\rho(f,g) = \rho(g,f)$, and the triangle inequality holds ($\rho(f,h) \leq \rho(f,g) + \rho(g,h) \ \forall f,g,h \in \mathcal{S}$). Define a zero-element (θ) of a set $\mathcal{S}$ by $f + \theta = f$ and multiplication by the scalar 0 equivalent to the zero vector. A *normed vector space* is a set $\mathcal{S}$ and a function *norm* $||\cdot|| : \mathcal{S} \to \mathbb{R}$ such that $||f|| \geq 0 \ \forall f \in \mathcal{S}$ (equality holds iff $f = \theta$), $||\alpha f|| = |\alpha| ||f|| \ \forall f \in \mathcal{S}, \alpha \in \mathbb{R}, \text{and} \, ||f+g|| \leq ||f|| + ||g|| \ \forall f,g \in \mathcal{S}$. For the next few definitions, assume $(\mathcal{S},\rho)$ is a metric space. $\{f_n\}_{n=0}^{\infty}$ ($f_n \in \mathcal{S} \ \forall n$) is *convergent* sequence (to f) if for every $\epsilon > 0 \ \exists N(\epsilon)$ s.t $\rho(f_n, f) < \epsilon \ \forall n \geq N(\epsilon)$. A *Cauchy* sequence is an equivalent way of specifying convergence in $\mathbb{R}$, and more generally is defined by an $N(\epsilon)$ (for a given $\epsilon > 0$) satisfying $\rho(f_n, f_m) < \epsilon \ \forall n, m \geq N(\epsilon)$. A metric space is *complete* if every Cauchy sequence in $\mathcal{S}$ is a convergent sequence in $\mathcal{S}$. Let $(\mathcal{S},\rho)$ be a complete metric space and $T : \mathcal{S} \to \mathcal{S}$, then T is a **contraction mapping** with modulus β if there exists $\beta \in [0,1)$ such that $\rho(Tf, Tg) \leq \beta\rho(f,g) \, \forall f,g \in \mathcal{S}$. Also note that for notation, the usual inequality relations with respect to functions, $f \geq g$ if $f(x) \geq g(x) \ \forall x \in X$. Finally, let $C(X)$ be the space of bounded continuous functions on X and $C'(X)$ be the space of bounded, continuous, and concave functions. Now we can state the results which collectively say something about our problem of interest

Theorem of the Maximum Assume $x \in X \subseteq \mathbb{R}^l, y \in Y \subseteq \mathbb{R}^m, f : X \times Y \to \mathbb{R}$ continuous, and $\Gamma : X \to Y$ compact valued and continuous. Define generalized value and policy set

$$V(x) = \max_{y \in \Gamma(x)} F(x,y) \ \ \underline{\textbf{and}} \ \ H(x) = \{y \in \Gamma(x) | f(x,y) = V(x)\}$$

Then $V(x)$ is continuous and $H(x)$ is non-empty, compact-valued, and *upper hemi-continuous* [1] (UHC).

Blackwell's Theorem* Let $T : \mathcal{F} \to \mathcal{F}$ be an operator defined on the metric space w.r.t $\mathcal{F}$ and the sup norm. Assume that T satisfies (monotonicity) $f \geq g \implies Tf \geq Tg$ and (discounting) $\forall f \in \mathcal{F} \, \text{and} \, c \in \mathbb{R}^+$, $\exists \beta \in [0,1)$ s.t $T(f+c) \leq Tf + \beta c$. Then T is a contraction mapping with modulus β

Contraction Mapping Theorem* Given a contraction mapping $T : \mathcal{S} \to \mathcal{S}$ with modulus β, if $(\mathcal{S},\rho)$ is a complete metric space, then there is a unique point $f \in \mathcal{S}$ s.t $Tf = f$ and for any $f_0 \in \mathcal{S}$, the infinite sequence governed by $f_n = Tf_{n-1}$ satisfies $\rho(f_n, f) \leq \beta^n (f_0, f) \ \forall n$.

Immediate Application to Value and Policy Functions

Now we can consider applications that leave the generalized paradigm. Recall (2), our operator of choice. Blackwell's theorem, which we can invoke because the Theorem of the Max tells us T maps a continuous function to a continuous function ($T : C(X) \to C(X)$), implies T is a contraction mapping. Then, the Contraction Mapping Theorem says given our operator T *any* guess of a value function $v_0 \in C(X)$, we get a sequence of value functions ($v_n = Tv_{n-1}$) that limit to the optimal "fixed point" that was referenced a few sections ago. We can also define $v_n = T^n v_0$. Then, given A2 and A3, our T has a unique fixed point v and for *any* initial guess $v_0 \in C(X)$, $||v_n - v|| \leq \beta^n ||v_0 - v||$. Given our boundedness assumptions, the RHS limits to 0 as $n \to \infty$, so $v_n \to v$. We have also shown that the policy correspondence, defined as the set of "future values" that are both feasible and satisfy the aforementioned fixed point, is compact and upper hemi-continuous. The rest of this subsection will be more formally stating and listing these results.

(Existence/Uniqueness of Value Function) Given A2 and A3, the operator from (2), $T : C(X) \to C(X)$, has a unique fixed point $v \in C(X)$. And for any $v_0 \in C(X)$, $||T^n v_0 - v||| \leq \beta^n ||v_0 - v||$ and the (specific) optimal policy correspondence, $\pi(x) = \{y \in \Gamma(x) | v(x) = F(x,y) + \beta v(y)\}$ is compact valued and UHC.

Recall the Sequence Problem vs. Functional Equation section, and some of the notation introduced in it. Namely, the FE formulation and feasible plans $\Pi(x_0)$. Then, implicitly using some of the theorems and math results we have discussed, we can state some highly relevant results a bit more concisely

[1] given a and open neighborhood V of $G(a)$ for a correspondence $G : A \to B$, $\exists$ a neighborhood U s.t $\forall x \in U, G(x) \subset V$

(Existence of Solutions) Given A1-A4,then there exists a unique and continuous bounded function $v : X \to \mathbb{R}$ satisfying (2), the FE. Moreover, for any $x_0 \in X$, an optimal plan $\mathbf{x}^* \in \Pi(x_0)$ exists

(Concavity of the Value Function) Given A1-A6, then the aforementioned unique v that satisfies the FE is strictly concave.

(Monotonicity of Value Function) Given A1-A4, A7, and A8, the the unique v satisfying the FE is strictly increasing in all of its arguments

(Social Planning Envelope Theorem*) Given A2,A3,A5,A6,and A9, if $x_0 \in \text{Int}(X)$ and $h(x_0) \in \text{Int}(X)$, then v is continuously differentiable at x_0 with $v_x(x_0) = F_x[x_0, h(x_0)]$.

(Existence of Optimal Policy Function) Given A1-A6, then there exists a unique optimal plan $\mathbf{x}^* \in \Pi(x_0)$ satisfying the FE for any $x_0 \in X$. Moreover the optimal plan can be expressed via $\pi : X \to X$, a continuous policy function such as $\pi(x^*_{t+1}) = x^*_{t+1}$

(Form of Policy Function) Given A2,A3,A5, and A6, we the aforementioned policy correspondence $\pi(x) = \{y \in \Gamma(x) | v(x) = F(x, y) + \beta v(y)\}$ is single-valued and continuous. Since this means its a set with one element, it can be re-written by $\pi(x) = \underset{y \in \Gamma(x)}{\text{argmax}}\ F(x, y) + \beta v(y)$

(Convergence to Policy Function) Recall $v_n = T v_{n-1}$. Let $\pi_n = \underset{y \in \Gamma(x)}{\text{argmax}}\ F(x, y) + \beta v_n(y)$. If v satisfies the FE, then given A2,A3,A5, and A6, $\pi_n \to \pi$

Deeper Policy Function Application:

Returning to our problem, using capital and consumption, the optimal policy function for capital is $h(k) = k'$. We have shown under relevant assumptions, the policy function is single-valued and continuous. We also want to have a policy function that is increasing and limits to the steady state both above and below (i.e if the input is below/above the steady state then the function returns something larger/smaller). Further, the monotonicity combined with the limiting conditions (approaching 0 and ∞), we know that a graphing of the policy function against k will cross the $45°$ line at some point, meaning the policy function satisfies the steady state ($k^* = h(k^*)$). First, under regularity conditions guaranteeing the FOC solution is in $\text{Int}([0, \overline{k}])$ for some $\overline{k} \in \mathbb{R}$, then we have $u_c[g(k) - h(k)] = \beta v_k[h(k)]$. Let k_0 and k_1 be feasible values of capital such that $k_1 > k_0$. We will prove $h(k_1) > h(k_0)$ by contradiction. Assume to the contrary. Then by the strict concavity of u and v

$$u_c[g(k) - h(k)] = \beta v_k[h(k)] \leq \beta v_k[h(k_1)] = u_c[g(k_1) - h(k_1)] \implies g(k_0) - h(k_0) \geq g(k_1) - h(k_1)$$

So since we assumed $h(k_0) > h(k_1)$, the last inequality implies $g(k_0) > g(k_1)$, which is a contradiction of the prior knowledge that g is strictly increasing. Next we will show the limit/trend conditions (with respect to the steady state). The strict concavity of v implies for any feasible k, we have $[k - h(k)]\{v_k(k) - v_k[h(k)]\} \leq 0$. By the Euler and envelope equations, this implies $[k - h(k)][u_c(c)g_k(k) - \beta^{-1}u_c(c)] \leq 0$. $u_c(\cdot)$ is strictly positive, so $[k - (h(k)][g_k(k) - \beta^{-1}] < 0$ for $k \neq h(k)$ because $g_k(k^*) = \beta^{-1}$. Since g is concave, $k^* > k \implies g_k(k) - \beta^{-1} > 0$, so combining the last two inequalities $k^* > k \implies h(k) > k$. The opposite holds when $k^* < k$. So now we have shown the *global stability* of the policy function: given a feasible starting capital, capital approaches the steady state from above if it starts above and below if it starts below.

We can similarly show that this implies the policy function for consumption has some similar properties. Let $k_0 < k_1$ and define the policy function for consumption by $c(k) = g(k) - h(k)$. Then from $g(k_0) - h(k_0) \leq g(k_1) - h(k_1)$ above (remember we did a proof by contradiction) and the concavity of the utility function, we have $c(k_0) < c(k_1)$. We can also see this from this concavity of the value function, and then extrapolate the result from a combination of the Euler, envelope, and concavity of the utility functions. Thus $k_0 < k_1 \implies c(k_0) < c(k_1)$, so the policy function for consumption is increasing. We also have the same global stability conditions, but the poof is not quite as intuitive. First, define $\mathcal{A}(k) = u_c[c(k)]$ and $\beta\mathcal{A}(k)g_k(k)$; these correspond to both sides of the Euler equation. Note that $k < k^* \implies \mathcal{A}(k) < \mathcal{B}(k)$ since this also implies $g_k(k) > \beta^{-1}$, and the opposite holds for $k > k^*$. The derivative of these objects are $\mathcal{A}_k(k) = u_{cc}[c(k)]c_k(k)$ and $\mathcal{B}_k(k) = \beta\mathcal{A}_k(k)g_k(k) + \beta\mathcal{A}(k)g_{kk}(k)$, so $\mathcal{A}_k(k), \mathcal{B}_k(k) < 0$. Therefore we know the original objects are strictly positive and decreasing. Further the Euler equation implies that $\mathcal{A}(k_0) = \mathcal{B}(k_1)$ for optimality. Consider a graph of $\mathcal{A}(k), \mathcal{B}(k)$ against capital. First, think about the $k_0 < k_1$ case. The y values will be equivalent (visualize a horizontal line) at the point at which the line $k = k_0$ intersects $\mathcal{A}(k)$ and the line $k = k_1$ intersects $\mathcal{B}(k)$. So obviously, to get the next value of $\mathcal{A}(k)$, you go down the $k = k_1$ until you hit the $\mathcal{A}(k)$ line, then travel horizontally (on the line $\mathcal{A}(k_1)$) to get the next value of $\mathcal{B}(k)$. If we do this process for every value of k, going from the $\mathcal{A}(k)$ curve to $\mathcal{A}(k)$, you will continue to "hit a wall", until you reach the steady state and cannot travel vertically down from the $\mathcal{B}(k)$ line because you are already on the other curve. More simply: $k_0 < k^* \implies k_t \leq k_{t+1} \leq k^*$. The dynamics are flipped when k_0 starts above the steady state, so we have the opposite: $k_0 > k^* \implies k_t \geq k_{t+1} \geq k^*$. Since the derivative of the policy function for capital is strictly positive, we also have $k_0 < k^* \implies c_t \leq c_{t+1} \leq c^*$ and $k_0 > k^* \implies c_t \geq c_{t+1} \geq c^*$

Transversality Condition

This is a part of a sufficient condition for an optimal plan. If a plan satisfies both the transversality condition plus the Euler equation, then it is an optimal plan (the converse is not necessarily true). Given A2,A3, A5, A8,A9, $X \subset \mathbb{R}^l_+$, and $x^*_{t+1} \in \text{Int}\,\Gamma[x^*_t]$ $(\forall t)$ the plan is optimal if it satisfies

$$F_y(x^*_t, x^*_{t+1}) + \beta F_x(x^*_{t+1}, x^*_{t+2}) = 0 \ \ \underline{\textbf{and}} \ \ \lim_{t\to\infty} \beta^t F_x(x^*_t, x^*_{t+1}) \cdot x^*_t = 0$$

,where the second equality is the transversality equation and the first is the combination of the Euler and envelope condition. We can think about this condition in the context of the phase diagrams: we must satisfy the FOCs and ensure that there will not be a "blow up" either in marginal utility or consumption value.

Ljungqvist-Sargent Approach

Instead of substituting out capital, which is the "Stokey&Lucas" method described in earlier sections, the "L-S" method includes all variables, separating them into categorizations of *control* and *state*. In the simple case, we can consider today's capital a state variable (this is what we take as given at period 0) and today's consumption as the control. Along with a policy function for consumption, this gives the form $v(k) = \max_c u(c) + \beta v(k') = \max_c u(c(k)) + \beta v(g(k) - c(k))$. Assuming differentiability, this yields

$$\begin{aligned} v_k(k) &= u_c[c(k)]c_k(k) + \beta v_k[g(k) - c(k)][g_k(k) - c_k(k)] \\ &= \{u_c[c(k)] - \beta v_k[g(k) - c(k)]\}c_k(k) + \beta v_k[g(k) - c(k)]g_k(k) \end{aligned}$$

However, from the relevant Euler equation $u_c(c) = \beta v_k[g(k) - c]$, the term on the left and therefore this implies that $v_k(k) = \beta v_k[g(k) - c(k)]g_k(k) = u_c(c)g_k(k)$. Note that this approach doesn't have an "envelope" condition, but we arrive at an equivalent result anyway.

L-S Envelope Condition*: This is more precisely the Benveniste-Scheinkman formula, which is an academic formalization of the envelope theorem. Make standard assumptions[2] like the ones for the more simple envelope conditions, and impose that $\pi(\cdot,\cdot) = x'$ is a more generalized policy function (i.e takes on more inputs), and we have a policy function for a second variable ($q(x) = u$, which is analogous to consumption). Then

$$V(x) = \max_u \{F(x,u) + \beta V\big[\pi(x,u)\big]\} \implies V'(x) = \frac{\partial F\big[x, q(x)\big]}{\partial x} + \beta \frac{\partial \pi\big[x, q(x)\big]}{\partial x} V'\big[\pi\big(x, q(x)\big)\big]$$

The result stems from the cancellation of relevant terms as a result of the FOC:

$$\frac{\partial F(x,u)}{\partial u} + \beta V'\big[\pi(x,u)\big]\frac{\partial \pi(x,u)}{\partial u} = 0$$

[2]See here for a formal proof

4 Closed Econ Neoclassical Model

Properties and Constraints: This model assumes a closed economy and closed economy. Since the economy is closed, the bond market does not have the ROW to supply demand, so it has to clear domestically ($b_t = 0$) at every period. Because of this clearing condition, bond prices are no longer exogenous because they depend on the clearing condition. We make the same no-leisure and $h_t = 1$ normalization, giving a resource constraint of $c_t + k_{t+1} - (1-\delta)k_t = f(k_t)$

Economy: Instead of the HH optimizing over its lifetime, we now consider that we have a "social planner" taking the initial input of capital as given and choosing an infinite sequence of capital (k_{t+1}) and consumption. This gives a Lagrangian of

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t (f(k_t) + (1-\delta)k_t - k_{t+1} - c_t)$$

Taking FOCs and substituting yields $\frac{u_c(c_t)}{u_c(c_{t+1})} = \beta[f_k(k_{k+1}) + 1 - \delta]$. So it follows that the steady state is again k^{NM}. From the increasing with diminishing returns and limit conditions, we know this steady state is unique, positive and finite. See Section 2.2 for investment intuition.

Transition Dynamics: Think about the dynamic programming section. We can apply the results about policy functions to say some useful things about this model. We have a well-defined policy function for capital that gives us an implicit policy function for consumption. Remember the L-S paradigm about "states" and "controls". The policy function takes an input today's capital, a state variable. This gives us an optimal choice for both tomorrow's capital, because the policy function for capital is defined as what will satisfy the unique fixed point where we plug our value function into our "refining operator" and get back the exact same function, and by extension today's consumption. Because of the monotonicity and ceiling/floor of the policy function for capital, it limits to a steady state, which means consumption does too, as long as the initial value given for capital is feasible. This implies this model has the characteristic of "global stability".

Now we will think about the transition dynamics in less general terms. Let's think more concretely about what "feasible capital level" means. Recall $g(k) = f(k) - (1-\delta)k$. So consumption, $c(k) = g(k) - h(k)$ reaches a minimum when all resources are put towards tomorrow's capital ($g(k) = h(k)$) and a maximum where we consume everything and don't leave resources for tomorrow ($h(k) = 0$). This gives "lifetime" bounds on capital. Obviously, you can always consume everything, giving a lower-bound on lifetime capital as 0. In the dynamic programming section, we left $\overline{k} \in \mathbb{R}$ as an unknown upper bound. But we can define this explicitly by thinking of making the decision to consume nothing in every period, in other words reaching a steady-state of $c = 0$. This would imply that $k = g(k)$. So define $\overline{k}$ such that $\overline{k} = \delta^{-1} f(\overline{k})$. Now we know that capital cannot exceed the bounds of $[0, \overline{k}]$.

Now to understand the actual period to period movement, consider the rewriting of the budget constraint:

$$c_t - (f(k_t) - \delta k_t) = k_t - k_{t+1} \therefore\ k_t \geq k_{t+1} \iff c_t \geq f(k_t) - \delta k_t \ \underline{\textbf{and}}\ k_t \leq k_{t+1} \iff c_t \leq f(k_t) - \delta k_t$$

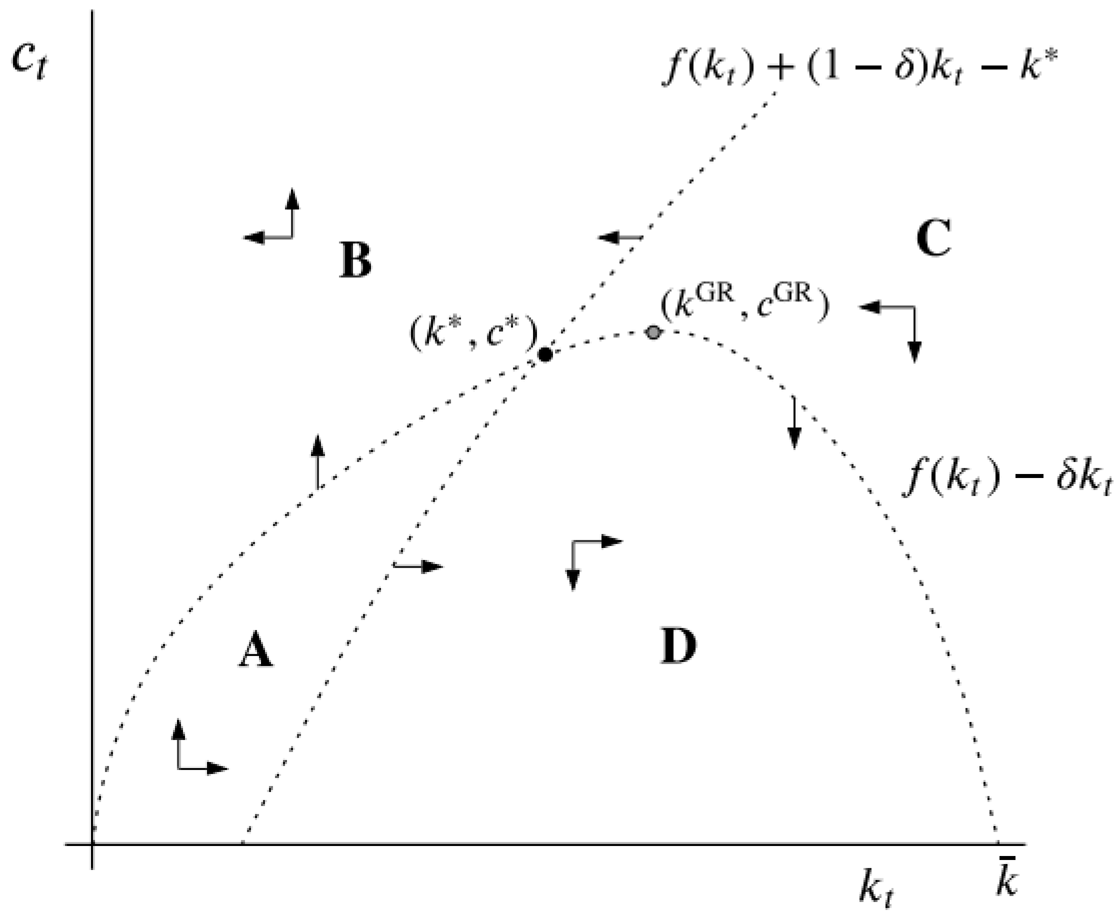
Recall $k^* = k^{\text{NM}} = f_k^{-1}(\delta + \rho)$. Again by the BC

$$k^* > k_{t+1} \iff k^* > f(k_t) + (1-\delta)k_t - c_t \iff c_t > f(k_t) + (1-\delta)k_t - k^*$$

Therefore

$$c_{t+1} \geq c_t \iff c_t \geq f(k_t) + (1-\delta)k_t - k^* \ \underline{\textbf{and}}\ c_{t+1} \leq c_t \iff c_t \leq f(k_t) + (1-\delta)k_t - k^*$$

These relationships/inequalities, combined with global stability, yield just one example of a precise way of describing the joint path of consumption and capital over time. We can visualize this in the form of a **phase diagram**, seen below, which relies on the major components of the above inequalities, namely how the magnitude of c_t relates to $f(k_t) - \delta k_t$ and $f(k_t) + (1-\delta)k_t - k^*$. Graphing these curves gives regions where certain conditions hold, telling us something about the evolution of capital and consumption



Condition	$f(k_t) - \delta k_t > c_t$	$f(k_t) + (1-\delta)k_t - k^* > c_t$	k	c	Optimality Feasible?
A	✓	**X**	↑	↑	✓
B	**X**	**X**	↓	↑	**X**
C	**X**	✓	↓	↓	✓
D	✓	✓	↑	↓	**X**

We will show that given $k_0 < k^*$, choosing c_0 from the policy function puts us in region **A** (indefinitely) and $k_0 > k^*$ puts us in region **C**. However, if the policy function is not followed initially, the saddle path will drift over into **B** or **D** and some of the nice properties (e.g $k_t < k^*$ implies montonistic increasing capital) no longer hold. Specifically (recall $c(k) = g(k) - h(k)$ is the policy function for consumption), if we pick $c_0 > c(k_0)$, then eventually the economy will enter region **B**, regardless of where we started. Similarly, picking $c_0 < c(k_0)$ makes region **D** an inevitability. So even though we may be tempted to use some of our results from the policy function and add an additional $\iff$ paired with some inequality relating k_t and k_{t+1} onto the end of a couple of these strings of math above, we cannot because it is possible that we will make a "bad" initial choice of consumption. The table above gives some intuition, including that capital and consumption should be "trending" in the same direction for optimality. Also notice for the two boundaries, one should be (optimally speaking) binding consumption from above and one should be binding below. Intuitively, we can think of this making an "optimality sandwich", for if both of the bounds are binding in the same direction, the path is unsustainable from an optimization perspective (i.e. we veer off to an undesirable point). *A more qualitative interpretation: if consumption is smaller than the gap between output and the amount of capital to be lost from depreciation, then capital is increasing (otherwise decreasing), and if consumption is larger than the gap between steady state capital and the sum of output and capital less depreciation, consumption is increasing (otherwise decreasing).* A simpler qualitative interpretation is $k_{t+1} = k_t$ along the quadratic curve (so going above/below produces an invariant inequality) and $c_{t+1} = c_t$ along the cubic curve.

We will go through each region and what could happen when we begin there. First, the non-trivial border of region cases are dealt with. Next, regions **B** and **D** will be discussed because the path is trapped there when it enters. This will make discussion for the two optimally feasible regions, **A** and **D**, much more straightforward.

Borders:

First, if a point lies on the boundary of **A** and **B** (but not at (k^*, c^*), this implies that $c_t = f(k_t) - \delta k \implies k_{t+1} = k_t$. Note that this implies $c_{t+1} > c_t$

$$c_{t+1} > c_t \iff u_c(c_{t+1}) < u_c(c_t) \iff \beta[f_k(k_{t+1}) + 1 - \delta > 1] \iff f_k(k_{t+1}) > \rho + \delta \iff k_{t+1} < k^*$$

So clearly the point changes vertically and not horizontally, thereby moving off the boundary into **B**, as implied in the diagram. Instead, if a point is on the boundary (non-steady state) of **B** and **C**, then $c_t = f(k_t) + (1-\delta)k_t - k^* \implies k_{t+1} = k^* < k_t$. If we go back to the string of math above, this also implies that $c_{t+1} = c_t$, so the point moves horizontally but not vertically into **B**. The other cases follow very similarly. The above math shows that we have for the non-steady state border (so $k_t > k^*$) of **C** and **D**

$$c_t = f(k_t) - \delta \implies k_t = k_{t+1} \implies k_{t+1} > k^* \implies c_{t+1} < c_t$$

Thus, the point moves down (no horizontal change) into **D**. Similarly, on the border of **A** and **D**, flipping the logic from what happened with regions **B** means that we again have $k_{t+1} = k^*$ and $c_{t+1} = c_t$, but this time $k_t > k^*$, so the point will be moving to the right with no vertical change, landing it in **D**. *Region **B***:

If we consider (k_t, c_t) lying in the interior of **B**, then $c_{t+1} > c_t$ from the above result that began with a main premise of **B**: $c_t > f(k_t) + (1-\delta)k_t - k^*$. This also implies $k_{t+1} < k^*$ from the math in the border paragraph. We also know from the other defining characteristic of **B** and the math above: $c_t > f(k_t) - \delta k_t \implies k_{t+1} < k_t$. As alluded to earlier, we should note that now all the norms we have for the policy function no longer hold; earlier we would expect capital to be increasing monotonically if $k_{t+1} < k^*$, but this logic assumes we are following the policy function from $t = 0$. Further, we know from the BC that the difference between today and tomorrow's capital stock is equal to the distance between c_t and $f(k_t) - \delta k_t$. Since capital is decreasing and consumption is increasing, we know this difference is increasing (because of the monotonicity of $f(\cdot)$), so the magnitude of tomorrow's capital stock is plunging at a faster and faster rate. Therefore, eventually we will get k_{t+1} 0 or negative. This means this regions yields an infeasible path since at some point we will not be able to solve for tomorrow's consumption. Implicit in this analysis is that once a path is in the interior of **B** it cannot leave.

*Region **D***:

Consider (k_t, c_t) lying in the interior of **D**. We know $k_t < \bar{k}$, but define an object $\bar{k}_t$ by $c_t = f(\bar{k}_t) - \delta \bar{k}_t$, in other words the value of capital that will lie on the $f(k_t) - \delta k_t$ curve/boundary, given the value of today's consumption (these two constructs are related; we will show, with respect to region **D**, $\bar{k}_t \to \bar{k}$ and $c_t = 0$). Since **D** has the defining characteristic that $c_t < f(k_t) + (1-\delta)k_t - k^*$, it follows that $\bar{k}_t > k_t$ in this region because it will have to go "to the right" to find the $f(k_t) - \delta k_t$ boundary. We also have

$$f(\bar{k}_t) + (1-\delta)\bar{k}_t > f(k_t) + (1-\delta)k_t \implies c_t + \bar{k}_t > f(k_t) + (1-\delta)k_t \implies \bar{k}_t > k_{t+1}$$

since $f(k_t) + (1-\delta)k_t$ is clearly strictly increasing and $k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$. $\bar{k}_t$ is bounded above by $\bar{k}$ because we can't have a negative value for consumption. Further, we know (from the math above) that consumption is decreasing and capital is increasing, so $\bar{k}_t$, the upper bound on k_{t+1} in **D**, has limit $\bar{k}$ (the point in which we cannot possibly increase the capital stock further). So it should follow intuitively that a path will limit to this ceiling. Suppose you instead thought it would limit to a point $\left(\widehat{k}, \widehat{c}\right)$, where $\widehat{c} \geq 0$ and $k^* < \widehat{k} < \bar{k}$ (we know it would have this lower bound since we already demonstrated $c_t < f(k_t) - \delta k_t$ implies a monotonistic increase in capital). Then it would follow that for any $\epsilon > 0 \ \exists T$ s.t $|c_{T+1} - c_T| < \epsilon$, meaning $k_{t+1} \to k^*$, which is a contradiction[3]. Further, because the boundaries implicitly impose that any path in the interior of **D** has decreasing consumption and increasing capital, a path that enters **D** cannot leave.

[3]This contradiction doesn't exist for $\bar{k}$ case because the Euler equation (i.e marginal rate of substitution) goes to 0

*Region **A***:

If a point begins in **A**, it cannot move into **C** because

$$c_t > f(k_t) + (1-\delta)k_t - k^* \implies k_{t+1} = f(k_t) + (1-\delta)k_t - c_t < k^*$$

so the remaining possibilities are obviously stay in **A** or move into **B** or **D**. If we follow the policy function, the path cannot escape **A** because in order to leave **A** it would have to violate the optimality conditions that are imposed by the policy function. If we choose to ignore the policy function, even for a period, the path will veer off into another region. To see this consider a choice of consumption $c_0 > c(k_0)$ (i.e. above optimal choice). Note that because capital is a state variable, and choosing tomorrow's capital determines what it will be, our decision to choose a level of capital above the steady state implies we will have $k_1 \neq h(k_0)$. We now can use some similar logic to the $c_{t+1} > c_t$ case:

$$c_0 > c(k_0) \implies u_c(c_0) < u_c(c(k_0)) \implies k_1 < h(k_0) \implies \beta[f_k(k_1)+1-\delta] > \beta[f_k(h(k_0)+1-\delta] \implies u_c(c_1) < u_c(c(k_1) \implies c_1 > c(h(k_0))$$

since $h(k_0) = f(k_0) + (1-\delta)k_0 - c(k_0), k_1 = f(k_0) + (1-\delta)k_0 - c_0,$ and $u_c(c_t) = u_c(c_{t-1})/\beta[f_k(k_t) + 1 - \delta]$. Implicit in this formulation is that even if we pick $c_1 = c(k_1)$, the next periods consumption will still be above what would be the optimal choice. Now consider generalizing this logic to a case where we randomly pick, not necessarily at date 0, a level of consumption above what the policy function tells us. That means the next periods capital will be below the policy function's choice, and this creates a never-ending cycle meaning we will never be able to get "back on track" to the path that would have happened with not deviating from the policy function. The same happens for consumption: because we are bound by the BC and evolution of capital equation, our single deviation prevents from rejoining the initial policy function path. In other words, the path will always lie above (visually, since consumption is on the y axis) the policy function path. Because of how the area region **A** shrinks and capital is growing (i.e moving right towards the steady state), we know that the margin of difference will eventually lead to the path leaving **A** and going into **B**. The case where we chose a below optimal consumption level (e.g. $c_0 < c(k_0)$) should follow pretty easily: we can just flip all the inequalities from above. So in this case, the path would stay below the policy function path and eventually leave **A** and go into **D**. Another way we can think about this is with the transversality condition from the last section. We know that if a path limits to the steady state pair, it satisfies the transversality condition. So since the paths can't limit to the steady state, they can't be optimal paths (also violates a notion of uniqueness). Put differently, any path able to stay in **A** is an (the) optimal path.

*Region **C***:

This follows very closely from what we saw with **A**. Except now we have an initial capital greater than the steady state. Again, we have that a path from **C** can't go into **A**, and the only way it will leave **C** is if a non-optimal path is chosen. If we choose $c_0 > c(k_0)$, the path will always lie above the optimal path, and as capital (and the area of region **C**) shrinks this means that the path will eventually cross into **B**. Similarly, $c_0 < c(k_0)$ implies the path will always lie below and the saddle path will eventually cross into **D**. In order to see this explicitly, simply go to the **A** paragraph and substitute out the notation properly. We can also again look to the transversality condition to confirm the notions of divergence and uniqueness.

Conclusion:

There are some important things to takeaway from this rigorous discussion. First, it's important to note the nuance of what all the different paths being discussed represent. We assume that the social planner, by following the policy function, will always pick the correct path. But if we (an outsider) is trying to emulate the decision of the social planner, say through a coding exercise where a sequence of consumption and capital is generated given an initial level of capital and an initial "guess" for consumption, we may pick wrong! Continuing with the coding example, we can include "if" conditions to see if we are veering off into the infeasible regions and then subsequently revising the initial guess of consumption. After so many times of violating the if conditions, the optimal path will eventually be found. Another important takeaway is that there are infinite possible initial pairs (k_0, c_0), each of which can be the beginning of a sequence satisfying Euler equation $\forall t$. The implication is that the Euler equation alone is not sufficient to guarantee that we are on the optimal path. This exercise has shown why it follows that if the transversality condition is also satisfied, *then* we have found the unique optimal path (unique to the initial k_0).

5 Competitive Equilibrium

Simple Example: Consider the "Robinson Crusoe" model where an agent lives one period. Now we don't make the no leisure assumption $\ell = 1 - h$. If we introduce technology to the model and substitute out leisure of the utility function, the we get F.O.C of $u_c(\cdot) = \lambda, u_\ell(\cdot) = \lambda A F_h(\cdot), \text{and } c = AF(\cdot)$. Now let's decentralize this simple model: where the supply and demand sides operate "autonomously" (i.e. take the behavior of the other side as given), where households own capital stock and rent to firms, sells work to firms, and maximizes utility and firms maximize profits subject to the economic conditions. More formally

$$\textbf{Firm:} \qquad \pi = \max_{k^d, h^d} AF(k^d, h^d) - r_k k^d - wh^d \;\; \text{s.t } \; r_k \text{ and } w \text{ given}$$

$$\textbf{HH:} \qquad \underset{c, h^s, k^s}{u(c, 1 - h^s)} \;\; \text{s.t } \; r_k, w, \text{and } \pi \text{ given and } c \in [0, wh^s + r_k k^s + \pi], k^s \in [0, k], h^s \in [0, 1]$$

where, to be explicit, (recall this a one period example) k is the capital stock for the entire economy, firms do not set wages, households do not set the rental rate, and households see their choices as not affecting profit. These assumptions will soon be generalized to more complex models, and you can think of them as implicit market clearing conditions: for instance, households have no reason to not rent out all of their capital in a situation where they only live one period. This gives a structure to completely define a **competitive equilibrium** (CE), where our definitions will slowly get more complex as we build our models off of different and less restrictive assumptions.

<u>*CE (simple)*</u>:
A competitive equilibrium is a set of allocations c, h^d, h^s, k^d, k^s, and a set of prices $w \text{ and } r_k$ s.t

- Taking $r_k, w, \text{and } \pi$ as given, the HH consumption demand and labor/capital supply is $c, h^s, \text{and } k^s$
- Taking $w \text{ and } r_k$ as given, the firm's output and labor/capital demand are $y, h^d, \text{and } k^d$
- Markets Clear: $y = c, h^s = h^d, \text{and } k^s = k^d$

This competitive equilibrium can be characterized further by looking at the optimality conditions. The firm's FOCs give the classic result that $AF_k(\cdot) = r_k \text{ and } AF_h(\cdot) = w$, which from Euler's theorem implies that $\pi = 0$. The HH FOCs yield a similar result to what we saw in the "centralized model" (go back and replace $AF(\cdot)$ with wh_s in the BC) and show $u_\ell(\cdot)/u_c(\cdot) = w$. We can impose market clearing conditions to simplify these results even further, finding that one of the market clearing conditions will be redundant (Walrus' law) because it will be implicitly "left over" after everything else is imposed.

Simple Dynamic Example: Consider a setup with exogenous lifetime income and no disutility/extraneous incentives involved with consumption ($c_t = y_t$). Now let's introduce "time 0" structure: the consumption good is widgets and household buy/sell claims to widgets for their lifetime (to be delivered at specified future dates) all in period 0. Prices are strictly relative – there is an "arbitrary unit of account"[4] (per widget) for each time period. This gives a BC of $\sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} p_t y_t$. Given a classic time-preference utility setup ($\sum_{t=0}^{\infty} \beta^t u(c_t)$) this yields a FOC of $\beta^t u_c(c_t) = \lambda p_t \implies p_{t+1} = p_t \beta u_c(c_{t+1})/u_c(c_t)$ considering the arbitrary unit of account assumption (we can set $p_0 = 0$, for example, since λ is indeterminate/prices are non-unique).

Now instead, consider a "sequential" structure: instead of all trade occurring at time 0, markets meet each period and households trade in a per-period, zero coupon bond market. So a HH chooses an infinite $\{c_t, b_{t+1}\}$. As previously there will need to be a "no ponzi" condition so the household does not die with debt (unconstrained, they would have an incentive to borrow an arbitrarily large amount and never repay). We can derive the condition using the previously seen iterative substitution within the budget constraint

$$c_t + q_t b_{t+1} = y_t + b_t \implies q_0 q_1 q_2 b_2 = y_0 + b0 - c_0 + q_0(y_1 - c_1) + q_0 q_1 (c_2 - y_2) \implies \cdots \implies \text{we need } \lim_{T \to \infty} b_T \prod_0^T q_t \geq 0$$

[4]An analogous example would be a strictly digital currency (that doesn't even have a fiat paper backing)

CE (simple,dynamic, sequential):
A set of quantities $\{c_t, b_{t+1}\}_{t=0}^{\infty}$ and bond prices $\{q_t\}_{t=0}^{\infty}$ s.t

- Taking $\{q_t\}_{t=0}^{\infty}$ as given, the HH consumption/bond choices are given by $\{c_t, b_{t+1}\}_{t=0}^{\infty}$
- Markets Clear: $c_t = y_t \text{ and } b_t = 0$

Again, we can further characterize this setup by looking at its optimality conditions, which are $\beta^t u_c(c_t) = \lambda_t p_t \text{ and } q_t \lambda_t = \lambda_{t+1} \implies q_t = \beta u_c(c_{t+1})/u_c(c_t)$. If we think about q_t as a proxy of relative price (since bonds bought in the previous period - b_t - payout at a rate of one per unit, whereas bonds bought in this period - b_{t+1} payout at a rate of q_t), it's easy to think about the time 0 and sequential setups as yielding functionally equivalent price solutions.

Decentralized Neoclassical: Adding "no leisure" and extending some of the assumptions from the simple model, we impose that households own all of the factors of production and have an initially equal distribution of capital and firm ownership. First, again consider a time 0 structure: all trade takes place in time 0, p_t is the price of widgets at time t that is set at time 0 (denominated in an arbitrary unit of account), $r_{k,t}$ is the rental rate of capital in terms of time t goods, and w_t is the price of labor denominated in time t goods (i.e a real wage). With both sides taking $k_0, p_t.w_t, r_{k,t}$ given $\forall\, t$, this yields

Firm:
$$\Pi = \sum_{t=0}^{\infty} p_t[AF(k_t^d, h_t^d) - r_{k,t}k_t^d - w_t h_t^d]$$

HH:
$$\sum_{t=0}^{\infty} \beta^t u(c_t) \;\text{ s.t }\; \sum_{t=0}^{\infty} p_t[c_t + k_{t+1} - (1-\delta)k_t] \le \sum_{t=0}^{\infty} p_t[r_{k,t}k_t^s + w_t h_t^s] \text{ and } c_t \ge 0, k_{t+1} \ge 0, k_y^s \in [0, k_t], h^s \in [0,1] \;\; \forall\, t$$

where Π is taken as given by the HH, implying if its non-zero then there is no credit/debit to account for via firm ownership in the BC, and the Firm and HH choice variables are defined intrinsically, as shown below.

CE (neoclassical, time 0):
Given k_0, an infinite set of prices $\{p_t, r_{k,t}, w_t\}$ and allocations $\{h_t^d, h_t^s, k_t^d, k_t^s, k_{t+1}\}$ s.t

- Given $\{p_t, r_{k,t}, w_t\}$, $\{k_t^d, h_t^d\}$ solves the firm's problem
- Given $\{p_t, r_{k,t}, w_t\} \text{ and } \Pi$, $\{c_t, k_{t+1}, h_t^s, h_t^s\}$ solves the HH problem
- Markets clear: $k_t^s = k_t^d, h_t^s = h_t^d, \text{and } c_t + k_{t+1} - (1-\delta)k_t = F(k_t^d, h_t^d) \;\; \forall\, t$

Once more, this setup yields the usual factor price equations of $w_t = F_h(\cdot) \text{ and } r_{k,t} = F_k(\cdot)$, implying zero profits. Imposing market clearing and no leisure, we can combine the FOCs for the household to get $\lambda p_t = \beta^t u_c(c_t) \text{ and } \lambda p_t = \lambda p_{t+1}(r_{k,t+1} + 1 - \delta) \implies \beta u_c(c_{t+1})[F_k(k_{t+1}, 1) + 1 - \delta] = u_c(c_t)$. We also see that the ratio of (temporally adjacent) prices is equal to the discounted ratio of marginal utilities.

A sequential setup provides an opportunity for a recursive structure, but as implied earlier in this section and in the dynamic programming section, the solution will be functionally equivalent, in this case because it's a "static" problem. In order to emulate the fluctuations of the balance between income and expenditures across time, a zero coupon bond market needs to be introduced. Because factor prices have been continually shown to be functions of the capital stock, in order to keep with the paradigm that factor prices are known to agents impose that HH have knowledge of the capital stock $k' = \Re(k)$. We still consider the HH to be *atomistic*, believing its choices do not affect aggregate economic levels, but this time this includes prices, not just profit. This example is largely reflective of reality: a consumer, for instance, wouldn't think that boycotting McDonald's would lead to the downfall of the company, even though if they were "representative" (and was followed en masse) it would. But the HH's choice needs to be consistent with the aggregate. So define so notation: $K \text{ and } B$ represent the HH capital and bonds, with kb for the aggregate stock. Market clearing/closed economy implies $b = 0$.

Subject to the HH BC $q(k)B' = r_k K^S + w(k)H^s + \pi(k) + B - C - I$ and evolution of capital $K' = (1-\delta)K + I$

$$V(K,B,k) = \max_{C,I,K^s,H^s} u(C) + \beta V(K',B',k')$$

Note that this Bellman conforms to the L-S paradigm. The firm seeks to maximize

$$\pi(k) = \Pi = F(K^d, H^d) - r_k(k)K^d - w(k)H^d$$

Let $\cdot$ (for this setup only) be (K,B,k). Define relevant functions $V(\cdot), \mathcal{C}(\cdot), \mathcal{I}(\cdot), \mathcal{K}^s(\cdot), \mathcal{H}^s(\cdot), \mathcal{K}^d(k)$, and $\mathcal{H}^d(k)$. Define $\mathcal{K}(\cdot) = (1-\delta)K + \mathcal{I}(K,B,k)$ and $q(k)\mathcal{B}(\cdot) = r_k(k)\mathcal{K}^s(\cdot) + w(k)\mathcal{H}^s(\cdot) + \pi(k) + B - \mathcal{C}(\cdot) - \mathcal{I}(\cdot)$
CE (neoclassical, recursive, sequential):
A set of functions $V(\cdot), \mathcal{C}(\cdot), \mathcal{I}(\cdot), \mathcal{K}^s(\cdot), \mathcal{H}^s(\cdot), \mathcal{K}^d(k)$, and $\mathcal{H}^d(k), \Re(k), q(k), r_k(k), w(k)$, and $\pi(k)$ s.t

- V satisfies the HH Bellmamn
- Taking $\Re(k), q(k), r_k(k), w(k)$, and $\pi(k)$ as given, the HH choices of C, I, K^s, and H^s follow $\mathcal{C}, \mathcal{I}, \mathcal{K}^s$, and $\mathcal{H}^s$
- Taking $r_k(k)$ and $w(k)$ as given, the Firm's choices of K^d and H^d follow $\mathcal{K}^d$ and $\mathcal{H}^d$
- Markets Clear: $\mathcal{K}(k,0,k) = 0, \mathcal{K}(k,0,k) = \Re(k), \mathcal{K}^s(k,0,k) = \mathcal{K}^d(k)$, and $\mathcal{H}^s(k,0,k)$

Firm FOCs yield the usual $r_k(k) = F_k$ and $w(k) = F_h$. With market clearing conditions slightly simplifying the problem, we get HH FOCs of $u_c(C) = q(k)^{-1}\beta V_B \cdot'$ and $V_K(\cdot') = q(k)^{-1}V_B(\cdot')$. We also have envelope conditions $V_k(\cdot) = \beta V_K(\cdot')(1-\delta) + r_k(k)q(k)^{-1}\beta V_B(\cdot')$ and $V_B(\cdot) = q(k)^{-1}\beta V_B(\cdot')$. Combining these conditions yields the usual Euler condition seen in past setups of $q(k) = \beta u_c(C')/u_c(C)$. We can also substitute out the value function for bonds with the value function for capital relation, and then substitute out the value function for capital with the marginal utility relation, yielding a different form of $u_c(C) = \beta u_c(C')(r_k(k') + 1 - \delta)$. This model can also be simplified even further using the market clearing conditions on the BC, with the capital and labor functions respectively becoming k and 1, this implies profit is 0 from Euler's theorem. With a representative HH holding no bonds and $\mathcal{I}(\cdot) = \Re(k) - (1-\delta)k$, we get a BC of $\mathcal{C}(\cdot) + \Re(k) - (1-\delta)k = F(k,1)$. Combined with $u_c[\mathcal{C}(k,0,k)] = \beta u_c(\mathcal{C}[\Re(k), 0, \Re(k)])(F_k(\Re(k),1) + 1 - \delta)$ from the Euler/Envelope results, we now have a formulation that pins down solutions for capital and tomorrow's consumption. We also very clearly can pin down bond prices using the two results that related marginal utility.

6 Stochastic Models

A Prelude on Time Series Stuff

First Order: Consider an AR(1) model $y_t = \lambda y_{t-1} + w_t \implies (1-\lambda)y_t = w_t$. We also have

$$y_t = \lambda(\lambda y_{t-2} + w_{t-1} + w_t = \lambda^2 y_{t-2} + w_t + \lambda w_{t-1} \implies \cdots \implies (1-(\lambda L)^k)y_t = \sum_{j=0}^{k}(\lambda L)^j w_t$$

So if $(\lambda L)^k y_t$ limits to 0 and the RHS is finite, $y_t = \sum_[j = 0]^\infty (\lambda L)^j w_t$. So combining all of these results, we can conjecture that $(1-\lambda L)^{-1} = 1 + \lambda L + (\lambda L)^2 + \dots$. But notice that it's also possible to similarly derive a "forward looking" representation by

$$y_t = \lambda^{-2} y_{t+2} - \lambda^{-1} w_{t+1} - \lambda^{-2} w_{t+2} \implies \cdots \implies (1-(\lambda L)^{-k})y_t = -\sum_{j=0}^{k}(\lambda L)^{-j} w_t$$

where $y_{t+1} = F y_t = L^{-1} y_t$. So imposing some similar limit conditions as above, we could also conjecture that $(1-\lambda L)^{-1} = -(\lambda L)^{-1} - (\lambda L)^{-2} + \dots$. So now we have two different forms. Further complicating things is that our limit conditions imply that we could also write the solution $y_t = (1-\lambda L)^{-1} w_t + \lambda^t c$ for *any* $c \in \mathbb{R}$. To help establish some guidance, first assume y_0 is given, that all realizations of y_t and w_t are real valued, and $|\lambda| \neq 1$. Finally, impose that $c = 0$. Then for $|\lambda| < 1$ we should use a backward-looking solution (first derivation), and for $|\lambda| > 1$ a forward-looking solution.

Example (Solow): Consider the Solow model with Cobb-Douglas production $k_{t+1} = (1-\delta)k_t + sA_t k_t^\alpha$. We have a steady state of $k = (sA/\delta)^{1/(1-\alpha)}$. By a first order Taylor expansion (equality assumed)

$$(k_{t+1-k} = (1-\delta)(k_t - k) + sk^\alpha (A_t - A) + \alpha s A k^{\alpha-1}(k_t - k) \implies \widehat{k}_{t+1} = (1-\delta+\alpha\delta)\widehat{k}_t + \delta \widehat{A}_t$$

since $\delta = sAk^{\alpha-1}$. Suppose further that $A_t = A \implies \widehat{A}_t = 0$. Then since there is no deterministic variable, it would imply a general solution is $\widehat{k}_t = (1-\delta+\alpha\delta)^t c$. We would want this to hold at $t = 0$, so that gives $c = \widehat{k}_0$. Supposed instead we considered each realization of A_t to be in some compact set. Since $1-\delta+\alpha\delta < 1$, we can impose a backward looking solution of $\widehat{k}_{t+1} = \delta \sum_{j=0}^\infty (a-\delta+\alpha\delta)^j L^j \widehat{A}_t$.

Second Order: Now consider an AR(2) process $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t$. Thus

$$(1-\phi_1 L - \phi_2 L^2)y_t = w_t \implies (1-\lambda_1 L)(1-\lambda_2 L)y_t = w_t$$

by a quadratic decomposition. Clearly, we need $\lambda_1\lambda_2 = -\phi_2$ and $\lambda_1 + \lambda_2 = \phi_1$, so both λ_1, λ_2 must satisfy

$$\lambda^2 - \phi_1\lambda - \phi_2 = 0 \implies \lambda = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

from the quadratic formula. This also implies we have inverse roots by $1 - \phi z - \phi_2 z^2$.
We will usually deal with the distinct, real root case of $\phi_1^2 + 4\phi_2 > 0$.

Example (Neoclassical): FOCs yield $(f_k(k_{t+1}) + 1 - \delta)^{-1} = \beta u_c(c_{t+1})/u_c(c_t)$ and $c_t = f(k_t) + (1-\delta)k_t - k_{t+1}$. This steady state therefore is the pair (k, c) that satisfies $1 = \beta(f_k(k) + 1 - \delta)$ and $c = f(k) - \delta k$.

$$u_{cc}(c)(c_t - c) = \beta u_{cc}(c)[f_k(k)+1-\delta](c_{t+1}-c) + \beta u_c(c) f_{kk}(k)(k_{t+1}-k) \implies u_{cc}(c)c\widehat{c}_t = u_{cc}(c)c\widehat{c}_{t+1} + \beta u_c(c) f_{kk}(k)k\widehat{k}_{t+1}$$

$$\implies \qquad \widehat{c}_t = \widehat{c}_{t+1} + \frac{\mu}{\sigma}\widehat{k}_{t+1} \qquad \left(\sigma = \frac{-u_{cc}(c)c}{u_c(c)} > 0, \mu = -\beta f_{kk}(k)k > 0\right)$$

by a first order Taylor expansion. We can also do something similar for the resource constraint

$$(c_t - c) = f_k(k)(k_t - k) + (1-\delta)(k_t - k) - (k_{t+1} - k) \implies \widehat{c}_t = \frac{k}{c\beta}\widehat{k}_t - \frac{k}{c}\widehat{k}_{t+1}$$

since $\beta = (f_k(k) + 1 - \delta)$. Thus we can combine the two Taylor expansion results by

$$\frac{k}{c}(\beta^{-1}\widehat{k}_t - \widehat{k}_{t+1}) = \frac{k}{c}(\beta^{-1}\widehat{k}_{t+1} - \widehat{k}_{t+2}) + \frac{\mu}{\sigma}\widehat{k}_{t+1} \implies \exists \phi_1, \phi_2 \text{ s.t } (1-\phi_1 L - \phi_2 L^2)\widehat{k}_t = 0$$

where $\phi_1 = \beta^{-1} + 1 + (c\mu)/(k\sigma)$ and $\phi_2 = -\beta^{-1}$. We can represent the entire system by

$$\begin{pmatrix} \mu/\sigma & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{c}_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \beta^{-1} & -c/k \end{pmatrix} \begin{pmatrix} \widehat{k}_t \\ \widehat{c}_t \end{pmatrix} \implies \begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{c}_{t+1} \end{pmatrix} = \begin{pmatrix} \beta^{-1} & -c/k \\ -\mu/(\sigma\beta) & 1 + (\mu c)/(\sigma k) \end{pmatrix} \begin{pmatrix} \widehat{k}_t \\ \widehat{c}_t \end{pmatrix}$$

So it conforms to the general setup of $x_{t+1} = Mx_t$.
Recall also that since $(1 - \phi_1 L - \phi_2 L^2)\widehat{k}_t$ we have $(1 - \lambda_1 L)(1 - \lambda_2 L)\widehat{k}_t = 0$. By $|M - \lambda I| = 0$, *the eigen values of M correspond to the λ values we would get from this decomposition (see Hamilton Prop 1.1)*. And we can show that $|\lambda_1| < 1$ and $|\lambda_2| > 1$. To see this define $\theta = (\mu c)/(\sigma k) > 0$ and a function $\psi(\lambda) = (\beta^{-1} - \lambda)(1 + \theta - \lambda) - \theta\beta^{-1}$. The eigenvalues are found at setting this function to 0. The function is convex and quadratic, so by $\psi'(\lambda) = 2\lambda - (1 + \theta + \beta^{-1})$ $\psi(\lambda)$ is minimized at $\overline{\lambda} = .5(1 + \theta + \beta^{-1})$. Further, we have $\psi(\overline{\lambda}) < 0, \psi(0) > 0,$ and $\psi(1) < 0$. Therefore by the monotonicity of $\psi(\cdot)$ on the intervals on either side of $\overline{\lambda}$, $0 < \lambda_1 < 1 < \overline{\lambda} < \lambda_2$. So instead of writing the whole system in matrix form, we can just focus on capital by $\widehat{k}_t + c_1\lambda_1^t + c_2\lambda_2^t$ since we have a representation of $\widehat{k}_t$ that doesn't include another variable. By our assumptions (k_0 known and $\widehat{k}_t \to 0$), this gives us $\widehat{k}_0 = c_1 + c_2$, but since $c_2 \in \mathbb{R}$ and $\lambda_2^t \to \infty$ the limiting assumption for capital is only satisfied if $c_2 = 0 \implies c_1 = \widehat{k}_0$

Blanchard and Kahn Method: This method combines the processes of the two above approaches. Assume we have (like in the neoclassical example) a general form of $x_t = Mx_{t+1}$. Perform a decomposition $M = V\Lambda V^{-1}$, where V has columns corresponding to the eigenvectors of M in unit length and Λ is a diagonal matrix with the eigenvalues on the diagonal. Notationally, let v_{ij} denote the ij element of V and v^{ij} represent the ij element of V^{-1}. For this method to be useful, we need $v_{11} \neq 0$ (which is the case for the neoclassical model). Define $\widetilde{x}_t = V^{-1}x_t \implies \widetilde{x}_t = \Lambda\widetilde{x}_{t-1} \implies \widetilde{x}_{1,t} = \lambda_1\widetilde{x}_{1,t-1}$ and $\widetilde{x}_{2,t} = \lambda_2\widetilde{x}_{2,t-1} \implies \widetilde{x}_{1,t} = c_1\lambda_1^t$ and $\widetilde{x}_{2,t} = c_2\lambda_2^t$ since there is no other variable in either of the equations. Now we can "back out of" the tilde notation by reimposing its definition

$$x_t = V\widetilde{x}_t = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} c_1\lambda_1^t \\ c_2\lambda_2^t \end{pmatrix} = \begin{pmatrix} v_{11}c_1\lambda_1^t + v_{12}c_2\lambda_2^t \\ v_{21}c_1\lambda_1^t + v_{22}c_2\lambda_2^t \end{pmatrix}$$

which can be simplified with prior knowledge about the system
<u>*Shortcut:*</u> If you are previously familiar with the B-K method, then you can jump right to the step $\widetilde{x}_{1,t} = c_1\lambda_1^t$ and $\widetilde{x}_{2,t} = c_2\lambda_2^t$. If you know (like in the neoclassical model) $\lambda_2 > 1 \implies c_2 = 0 \implies \widetilde{x}_{2,t} = 0$. Then from the definition of $\widetilde{x}_{2,t}$ we have

$$0 = v^{21}x_{1,t} + v^{22}x_{2,t} \implies x_{2,t} = \frac{-v^{21}}{v^{22}}x_{1,t} = \frac{v_{21}}{v_{11}}x_{1,t}$$

by the definition of the 2×2 inverse. So we need $v^{22}, v_{11} \neq 0$ (these are equivalent). Then from $x_{1,t} = v_{11}c_1\lambda_1^t$, we get $c_1 = x_{1,0}/v_{11} \implies x_{1,t} = x_{1,0}\lambda^t = \lambda_1 x_{1,t-1}$.
Applying the neoclassical example, we get $\widehat{k}_t = \widehat{k}_0\lambda_1^t$ and $\widehat{c}_t = \widehat{k}_0\lambda_1^t v_{21}/v_{11}\lambda_1^t$

Time Series Processes: Let $\mathbf{y} = \{y_t\}_{t=-\infty}^{\infty}$. $\mathbf{y}$ is *stationary* if $f(y_t, \ldots, y_{t-k}) = f(y_{t+s}, \ldots, y_{t+s-k})$ $\forall\, s, k, t$. In other words, the distribution of a continuous sample of $\mathbf{y}$ must be identical to the distribution of any other continuous sample with the same length. This is obviously a very burdensome definition, so also consider that $\mathbf{y}$ is *covariance stationary if* $\mathbb{E}[y_t] = \mu, \mathrm{var}(y_t) = \gamma_0 \in \mathbb{R},$ and $\mathbb{E}[(y_t - \mu)(y_{t-s} - \mu)] = \gamma_s$ $\forall\, t, s$. An important nuance here is that the covariance only depends on the interval length, not where in the sample the interval is. Similarly, ϵ_t is a *white noise* process if $\mathbb{E}[\epsilon_t] = 0, \mathrm{var}(\epsilon_t) = \sigma^2 \in R,$ and $\mathrm{cov}(\epsilon_t, \epsilon_{t-s}) = 0$ $\forall\, t$ and $\forall\, s \neq 0$. We also have a class of processes that are defined, in some fashion, w.r.t a white noise process ϵ_t.
$AR(p) : y_t = \sum_{j=1}^{p} \phi_j y_{t-j} + \epsilon_t$ and $MA(q) : y_t = \sum_{j=1}^{q} \theta_j y_{t-j} + \epsilon_t$. This yields

Model	μ	γ_0	γ_s	ρ_s
MA(1)	0	$(1 + \theta_1^2)\sigma^2$	$\theta_1\sigma^2$, 0	$\theta_1/(1 + \theta_1^2)$, 0
MA(q)	0	$(1 + \theta_1^2 + \cdots + \theta_q^2)\sigma^2$	$(\theta_s + \theta_{s+1}\theta_1 + \cdots + \theta_q\theta_{q-s})\sigma^2$, 0	γ_s/γ_0
MA(∞)	0	$\sigma^2 \sum_{j=0}^{\infty} \theta_j^2$	$\sigma^2(\theta_s + \sum_{j=1}^{\infty} \theta_j\theta_{j+s})$	γ_s/γ_0
AR(1)	0	$\sigma^2/(1 - \phi_1^2)$	$\phi_1^s\gamma_0$	ϕ_1^s

where if there are two numbers for an order x process' metric they correspond to $s \leq x, s > x$.

Note that a AR(p) process can be represented as MA(∞), as long as the roots of $1 - \phi_1 z + \ldots \phi_p z = 0$ are outside the unit circle. More details in the appendix. Also any of the aforementioned definitions can be altered to include a constant, in which case $\mu \neq 0$

6.1 Stochastic Neoclassical

The fundamental assumption for this model is that $y_t = z_t f(k_t)$, where z_t can be thought of a stochastic shock analogous to the state of technology at a given time period. A trivial example would be maybe the power grid is really spotty, and your output is directly proportional to have much power capability you have at a given time period. Notationally, consider $z_t = z_t(s^t)$ and $s^t = (s_t, \ldots, s_0)$, implying the shock is a function of the history of a shock. We will assume that there are a finite number of possibilities for an event, in other words $s_t \in \{s[1], \ldots, s[N]\}$. We have $\pi_t(s^t)$ and $\pi_t(s^t|s^\tau)$ as the conditional and unconditional probabilities of observing a particular history, respectively.
So we have a general social planning problem of

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t(s^t)]\pi_t(s^t) \text{ s.t } c_t(s^t) = z_t(s^t) f[k_t(s^{t-1}] + (1-\delta)k_t(s^{t-1} - k_{t+1}(s^t)$$

With a LM that is also a function of the state, this yields FOCs of $\beta^t u_c[c_t(s_t)]\pi_t(s^t) = \lambda_t(s^t)$ and $\lambda_t(s^t) = \sum_{s^{t+1}|s^t} \lambda_{t+1}(s^{t+1})(z_{t+1}(s^{t+1}) f_k[k_{t+1}(s^t)] + 1 - \delta)$. Because $\pi_t(s^{t+1})/\pi_{t+1}(s^t) = \pi_{t+1}(s^{t+1}|s^t)$

$$u_c[c_t(s^t)] = \sum_{s^{t+1}|s^t} (z_{t+1}(s^{t+1}) f_k[k_{t+1}(s^t)] + 1 - \delta)\pi_{t+1}(s^{t+1}|s^t) \implies u_c(c_t) = \mathbb{E}_t \beta(z_{t+1} f_k(k_{t+1}) + 1 - \delta) u_c(c_{t+1})$$

The recursive social planning approach follow

$$V(k,s) = \max_c u(c) + \beta \sum_{s'} V(k', s')\pi(s'|s) \text{ s.t } k' = z(s)f(k) + (1-\delta)k - c$$

Here, an implicit "Markov Process" assumption is made where we consider that the realization of tomorrow's stochastic state only depends on the previous date's realization (as opposed to the entire history). This gives a FOC of $u_c(c) = \beta \sum_{s'} V_k(k', s')\pi(s'|s)$ and envelope $V_k(k,s) = (z(s) f_k(k) + 1 - \delta)\beta \sum_{s'} V_k(k', s')\pi(s'|s)$. Therefore, we have jointly $u_c(c) = \beta \sum_{s'} u_c(c')(z(s') f_k(k') + 1 - \delta)\pi(s'|s)$

We can also decentralize the model as we did in the previous section, where the prices are also functions of the history. Taking the prices and the stochastic shocks as given, the firm in the non-recursive setup solves

$$\Pi = \sum_{t=0} \sum_{s^t} p_t(s^t)[z_t(s^t) F[k_t^d(s^t), h_t^d(s^t)] - r_{k,t} k_t^d(s^t) - w_t(s^t) h_t^d(s^t)]$$

For the HH problem they take prices and profit as given and solve

$$\sum_{t=0} \sum_{s^t} \beta^t u[c_t(s^t)]\pi_t(s^t) \text{ s.t } \sum_{t=0} \sum_{s^t} p_t(s^t)[c_t(s^t) + k_{t+1}(s^t) - (1-\delta)k_t(s^{t-1})] = \sum_{t=0} \sum_{s^t} p_t(s^t)[r_{k,t} k_t^d(s^t) + w_t(s^t) h_t^d(s^t)] + \Pi$$

The firm FOCs are naturally $r_{k,t}(s^t) = z_t(s^t) F_k(\cdot)$ and $w_t(s^t) = z_t(s^t) F_h(\cdot)$, which again implies by Euler's theorem that profits are 0 at each date. Imposing market clearing conditions, the HH FOCs are $\beta u_c[c_t(s^t)]\pi_t(s^t) = p_t(s^t)$ and $p_t(s^t) = \sum_{s^{t+1}|s^t} p_{t+1}(s^{t+1})(r_{k,t+1}(s^{t+1} + 1 - \delta)$. Substituting in the firm FOCs and imposing market clearing on the BC, we get back the same solutions as the social planning setup.

We can also look at a decentralized, recursive model. The firm solves

$$\pi(k,s) = \max_{K^d, H^d} z(s) F(K^d, H^d) - r_k(k,s) K^d - w(k,s) H^d \text{ s.t } z(s), r_k(k,s), \text{and } w(k,s) \text{ given}$$

For the HH, to draw equivalence with the time 0 setup, we can consider a bond market where the bonds are proxies to hedge against the realizations of the stochastic shock (and thus only pay out if a certain shock occurs). To impose this structure explicitly, denote a $N \times 1$ vector $\widetilde{B}$, whose jth element represents claims against the jth possible state of the world. It's also useful to define a $N \times 1$ vector $X(s)$, which has a 1 at the $s = j$th element and 0 elsewhere. Also, as with the other setups, the bond vector represents the bonds purchased in the current time period and they pay out in the next. So we have a distinction that B is a current state ($B \cdot X(s)$ is what's paid out), $\widetilde{B}$ is a control variable, and $B' = \widetilde{B}$ is a future state variable. To emulate the "full information" aspect, consider a law of motion for aggregate capital $k' = \Re(k, s)$ known to the HH. The HH capital follows the usual $K' = (1 - \delta)K + I$. This yields a HH BC of

$$C + I + \widetilde{B} \cdot q = r_k(k, s)K^s + w(k, s)H^s + \pi(k, s) + B \cdot X(s)$$

So subject to the BC with $K' = (1 - \delta)K + I$ and $B' = \widetilde{B}$, the HH Bellman is

$$V(K, B, k, s) = \max_{C, K^s, H^s, I, \widetilde{B}} u(C) + \beta \sum_{s'} V(K', B', \Re(k, s), s')\pi(s'|s)$$

The firm FOCs and profit are fundamentally identical to the dynamic setup. For the HH, since consumption does not appear in the evolution of capital equation, we have two options: either substitute for investment and include it or consider a Lagrangian type optimization where a constraint (LM) is imposed with respect to the BC. We will show the implied Lagrange result. Imposing market clearing to eliminate H^s and K^s, we have FOCs of $u_c(C) = \lambda$, $\lambda = \beta \sum_{s'} V_k(\cdot')\pi(s'|s)$, and $q_j\lambda = \beta \sum_{s'} V_{B_j}(\cdot')\pi(s'|s) \;\; \forall j$ The envelope conditions are $V_k(\cdot) = r_k(k, s)\lambda + (1 - \delta)\beta \sum_{s'} V_K(\cdot')\pi(s'|s)$ and $V_{B_j}(\cdot) = X_j(s)\lambda \;\; \forall j$. Combining and summing over the probabilities

$$\sum_{s'} V_K(\cdot')\pi(s'|s) = \sum_{s'} \left[r_k(\cdot')\lambda' + (1 - \delta)\beta \sum_{s''} V_K(\cdot'')\pi(s''|s') \right] \pi(s'|s)$$

We can update the conditions using $\lambda = \beta \sum_{s'} \lambda'(r_k(\cdot') + 1 - \delta)\pi(s'|s)$, yielding

$$u_c(C) = \beta \sum_{s'} u_c(C')[r_k(\cdot') + 1 - \delta]\pi(s'|s)$$

The price, or ex-ante value, of the jth security is

$$q_j(k, s) = \beta \frac{u_c(C(k', \overline{0}, k', s[j]))}{u_c(C(k, \overline{0}, k, s))} \pi(s[j]|s)$$

where $\overline{0}$ denotes $B \to 0$. This condition is derived from the equities relating $u_c(C)$ and $V_{B_j}(\cdot')$ to λ

6.2 Stochastic Difference Equations

Consider an information set $\mathcal{I}_t$ that represents everything known to us at t. Then $\mathbb{E}_t y_{t+1} = \mathbb{E}[y_{t+1}|\mathcal{I}_t]$. Recall the first order difference equations. Imposing this structure we have $\mathbb{E}_t y_{t+1} = \lambda y_t + z_t$. This is also equivalent to $y_t = \lambda y_{t-1} + z_{t-1} + \epsilon_t$, where $\epsilon_t = y_t - \mathbb{E}_{t-1} y_t$. So naturally this can be rewritten as $y_t = (1 - \lambda L)^{-1}(z_{t-1} + \epsilon_t) + c\lambda^t$. Setting $c = 0$ is equivalent to assuming the model is in a neighborhood of the steady state in the growth model. If $|\lambda| < 1$, consider a backward looking (*direct*) solution by

$$y_t = \sum_{j=0}^{\infty} \lambda^j (z_{t-j-1} + \epsilon_{t-j}) = (1 + \lambda L + (\lambda L)^2 + \dots)Lz_t + (1 + \lambda L + (\lambda L)^2 + \dots)\epsilon_t$$

so let $z_t = \phi(L)v_{t-j}$, where v_t is a white noise process, and $av_t + u_t$, where u_t is also a white noise process. Then we have $y_t = (1 - \lambda L)^{-1}\psi(L)v_{t-1} + av_t + bu_t$ well-defined if the coefficient on v_{t-1} is square-summable, further implying that y_t is covariance stationary. We also have a forward looking solution $y_t = -\sum_{j=0}^{\infty} \lambda^{-(j+1)}(z_{t+j} + \epsilon_{t+1+j}$. We can make this well-defined by writing z_t as a factor of forecast errors

$$z_{t+1} = \mathbb{E}_t z_{t+1} + (z_{t+1} - \mathbb{E}_t[z_{t+1}]), z_{t+2} = \mathbb{E}_t z_{t+2} + (\mathbb{E}_{t+1} z_{t+2} - \mathbb{E}_t z_{t+2}) + (z_{t+2} - \mathbb{E}_{t+1} z_{t+2}), \dots$$

$$z_{t+s} = \mathbb{E}_t z_{t+s} + (\mathbb{E}_{t+1} z_{t+s} - \mathbb{E}_t z_{t+s}) + \dots + (\mathbb{E}_{t+s-1} z_{t+s} - \mathbb{E}_{t+s-2} z_{t+s}) + (z_{t+s} - \mathbb{E}_{t+s-1} z_{t+s})$$

So thus we can define $v_{t+s} = \sum_{j=0}^{\infty} \lambda^{-j}(\mathbb{E}_{t+s} - \mathbb{E}_{t+s-1})z_{t+s+j}$ then

$$\sum_{j=0}^{\infty} \lambda^{-j} z_{t+j} = \sum_{j=0}^{\infty} \lambda^{-j}\mathbb{E}_t z_{t+j} + \sum_{j=1}^{\infty} \lambda^{-j} v_{t+j}$$

Define $\epsilon_t = -\lambda^{-1} v_t$. By $\mathbb{E}_{t-1} v_t = 0$. we have

$$y_t = -\lambda^{-1} \sum_{j=0}^{\infty} \lambda^{-j}(z_{t+j} + \epsilon_{t+1+j}) = -\lambda^{-1} \sum_{j=0}^{\infty} \lambda^{-j}\mathbb{E}_t z_{t+j}$$

Note that these backward and forward formulations only work because of the equivalence of the new definitions to the $\epsilon_t = y_t - \mathbb{E}_{t-1} y_t$ we initially saw.

Example: (Linearized Stochastic Growth Model) We have the equations from the initial example

$$\widehat{c}_t = \widehat{c}_{t+1} + \frac{\mu}{\sigma}\widehat{k}_{t+1} \ \underline{\textbf{and}}\ \widehat{c}_t = \frac{k}{c\beta}\widehat{k}_t - \frac{k}{c}\widehat{k}_{t+1}$$

where $\sigma = \frac{-u_{cc}(c)c}{u_c(c)} > 0, \mu = -\beta f_{kk}(k)k > 0$. However, now we have a stochastic shock in front of the marginal product of capital and the output in the FOCs, as well as the expectation term. We can consider $\widehat{x} = \frac{dx}{x}$ as a proxy for the deviation variables, see more details in the appendix. So let $y = f(k)$ be output at the steady state, $z = 1$ be non-stochastic steady state, and $\mu_z = \beta f_k(k)$. Then we have a new system of

$$\widehat{c}_t = \mathbb{E}_t\widehat{c}_{t+1} + \frac{\mu}{\sigma}\widehat{k}_{t+1} - \frac{\mu_z}{\sigma}\mathbb{E}_t\widehat{z}_{t+1} \ \underline{\textbf{and}}\ \frac{c}{k}\widehat{c}_t = \frac{1}{\beta}\widehat{k}_t - \widehat{k}_{t+1} + \frac{y}{k}\widehat{z}_t$$

This gives a vectorized, entire system representation of

$$\mathbb{E}_t\begin{pmatrix}\widehat{k}_{t+1}\\ \widehat{c}_{t+1}\end{pmatrix} = \begin{pmatrix}\beta^{-1} & -c/k\\ -\mu/(\sigma\beta) & 1+(\mu c)/(\sigma k)\end{pmatrix}\begin{pmatrix}\widehat{k}_t\\ \widehat{c}_t\end{pmatrix} + \begin{pmatrix}0\\ \frac{\mu_z}{\sigma}\end{pmatrix}\mathbb{E}_t\widehat{z}_{t+1} + \begin{pmatrix}y/k\\ (-y\mu)/(k\sigma)\end{pmatrix}\widehat{z}_t \implies \mathbb{E}_t x_{t+1} = Mx_t + \zeta_t$$

because the coefficients on the "core terms" do are not affected because the log-linearization process separates everything. So we again have one eigen less than one and one greater. This gives us a system of $\mathbb{E}_t\widetilde{x}_{t+1} = \Lambda\widetilde{x}_t + \widetilde{z}_t$. So when we decompose this simplification into $\widetilde{x}_{1,t+1}$ and $\widetilde{x}_{2,t+1}$, the second variable (consumption) is problematic because we have a future realization on the RHS. We can use the shortcut in the B-K method from the time series subsection to say

$$\widetilde{x}_{2,t} = -\sum_{j=0}^{\infty} \lambda_2^{-(j+1)}\mathbb{E}_t\zeta_{2,t+j} = -\sum_{j=0}^{\infty} \lambda_2^{-(j+1)}(v^{21} \quad v^{22})\mathbb{E}_t\zeta_{t+j}$$

$$\widetilde{x}_{2,t} = v^{21}x_{1,t} + v^{22}x_{2,t} \implies x_{2,t} = \frac{-v^{21}}{v^{22}}x_{1,t} - \sum_{j=0}^{\infty} \lambda_2^{-(j+1)}(\tfrac{v^{21}}{v^{22}} \quad 1)\mathbb{E}_t\zeta_{t+j}$$

Now we have a well-defined policy function to select consumption at period t. Consider m_{ij} the ij element of M. Then we have

$$x_{1,t+1} = m_1 1 x_{1,t} + m_{12}x_{2,t} + \zeta_{1,t} = (m_{11} - m_{12}\frac{-v^{21}}{v^{22}})x_{1,t} + \frac{m_{12}}{v^{22}}\widetilde{x}_{2,t} + \zeta_{1,t}$$

We can also generate a "direct" solution using $\widetilde{x}_{1,t} = \sum_{j=0}^{\infty} \lambda_1^j(\widetilde{\zeta}_{1,t-j-1} + \epsilon_{t-j})$ and $\widetilde{x}_{2,t} = -\sum_{j=0}^{\infty} \lambda_1^{-(j+1)}\mathbb{E}_t\widetilde{\zeta}_{2,t+j}$, using the final results found above with $\epsilon_t = \widetilde{x}_{1,t} - \mathbb{E}_{t-1}\widetilde{x}_{1,t}$ white noise. We have $x_{1,t} = v_{11}\widetilde{x}_{1,t} + v_{12}\widetilde{x}_{2,t}$. Equivalently, $x_{1,t} = v_{11}\mathbb{E}_{t-1}\widetilde{x}_{1,t} + v_{11}\epsilon_t + v_{12}\mathbb{E}_{t-1}\widetilde{x}_{2,t} + v_{12}(\widetilde{x}_{2,t} - \mathbb{E}_{t-1}\widetilde{x}_{2,t})$, so $\epsilon_t = \frac{v_{12}}{v_{11}}(\widetilde{x}_{2,t} - \mathbb{E}_{t-1}\widetilde{x}_{2,t})$ and

$$x_{1,t} = v_{11}\mathbb{E}_{t-1}\widetilde{x}_{1,t} + v_{12}\mathbb{E}_{t-1}\widetilde{x}_{2,t} \ \underline{\textbf{and}}\ x_{2,t} = v_{21}\widetilde{x}_{1,t} + v_{22}\widetilde{x}_{2,t}$$

if we impose that when the "forecast error" terms will cancel each other out to in effect make $x_{1,t} \in \mathcal{I}_{t-1}$.

7 RBC Model

The real business cycle model is concerned with deriving a balanced growth path and representing the equation with respect to deviations from trends (usually through linearization)
Neoclassical model: $\mathbb{E}_0 \sum_{t=1}^{\infty} u(C_t, L_t); N_t + L_t = 1$; deterministic, labor enhancing growth $X_t = \gamma X_{t-1}; Y_t = A_t F(K_t, N_t X_t)$
Preferences $U(C_t, L_t) = \frac{1}{1-\sigma}\{[C_t v(L_t)]^{1-\sigma}\}$
Normalize variables with respect to X_t (except labor). Then because of CRS $y_t = A_t F(k_t, N_t)$ and $k_{t+1}\gamma = (1-\delta)k_t + i_t$. Let $\beta = b\gamma^{1-\sigma}$ Then we have

$$\mathbb{E}_0 \sum_{t=1}^{\infty} \beta^t u(c_t, 1-N_t) \text{ s.t } A_t F(k_t, N_t) \geq c_t + k_{t+1}\gamma - (1-\delta)k_t$$

$$\mathbb{L} = \sum_{t=1}^{\infty} \beta^t u(c_t, 1-N_t) + \sum_{t=1}^{\infty} \beta^t \lambda_t [A_t F(k_t, N_t) - \big(c_t + k_{t+1}\gamma - (1-\delta)k_t\big)]$$

Let $f_1(\cdot,\cdot)$ denote the derivative of f w.r.t its first argument. Then the FOCs are

$$u_1(c_t, 1-N_t) = \lambda_t; u_2(c_t, 1-N_t) = \lambda A_t F_2(k_t, N_t); \lambda_t\gamma = \beta \mathbb{E}_t \lambda_{t+1}(F_1(k_{t+1}, N_{t+1}) + 1 - \delta)$$

Some symbolic points to note: λ_t can be thought of as the shadow price of consumption; $\lambda_t A_t F_2(k_t, N_t)$ the marginal cost of labor, $\lambda_t \gamma$ the marginal cost of investment, and the last term the expected marginal benefit of investment.
A *deterministic steady state* (SS) is one where constant values are assumed. A *stochastic SS* is one where agents form a subjective probability distribution about future realizations (e.g. of the technology factor A_t).
Deterministic steady state: $AF_1(\frac{k}{N}, 1) = \frac{\gamma}{\beta} - 1 + \delta$; $\frac{c}{N} = F(\frac{k}{N}, 1) - (\delta + 1 - \gamma)\frac{k}{N}$; $\frac{u_2(c,L)}{u_1(c,L)} = AF_2(\frac{k}{N}, 1)$, where $F_1(k, N) = AF_1(\frac{k}{N}, 1)$ since $F_1(\cdot)$ is homogeneous degree 0.
Stochastic SS: Take output as a proportion of relative inputs. Then per the solow model, the "residual" is the technology factor; essentially accounting from the error from the functional form specifications. Specifically, let $\alpha_K = \frac{rK}{Y}$ and $\alpha_N = \frac{wN}{Y}$ be the factor shares (the ratio of the factor to output multiplied by the marginal product). Then the solow residual $S_t = \frac{Y}{K_t^{\alpha_K} N_t^{\alpha_N}}$. Since we impose $\alpha_K + \alpha_N = 1$ (i.e. constant returns to scale) we have $S_t = \frac{Y}{K_t^{\alpha} . N_t^{1-\alpha}} = A_t X_t^{1-\alpha}$. Taking logs, you can isolate the technology as the time variant component of the residual by detrending $\log(S_t) = \log(A_t) + (1-\alpha)(t\log(y) + \log(X_0))$.
Tune parameters to match the model. For instance let $r = \frac{\gamma}{\beta} - 1$ and match it to what is observed in the data for real interest rates. If we use a cobb-douglass production function and normalize $A = 1$ (with respect to the deterministic case), then the from MPK $\alpha = \frac{r+\delta}{Y/K}$. You can also directly look at the labor share of GDP and solve for α by setting labor share to $1-\alpha$.

First some general intuition. The wealth effect will be larger the more permanent a shock is. This is because the impact on the present discounted value of wealth is larger the more permanent the effect is (i.e. with a less permanent shock , the effect dissipates over time, so the increase in the discounted value is less). On the other hand, the substitution effect only looks at the initial shock itself; it's not affected by how long or how much the shock persists. Just that there had been a change in the present. So for a permanent shock, the wealth and substitution effects cancel out, but the less permanent a shock is the more the substitution effect dominates. Also note that $Y_t^d = C_t + I_t$, and that a large difference between Y_t^s and Y_t^d yields an incentive to increase the interest rate to stimulate the demand for goods. Below, at $t = 0$, initial interest rates are held constant. The arrows represent change relative to the steady state, with the exception of GE which shows change with respect to the initial effect Temporary, positive technology shock ($\mathbb{E}[\hat{A}_{t+1}] = 0$)

period	A_t	C_t	Y_t	I_t	N_t	R_t
$t = 0$, initial	$\uparrow$	$\uparrow$	$\uparrow\uparrow$	-	$\uparrow\uparrow$	$\downarrow$
$t = 0$, GE (response to $R_t \downarrow$)	-	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	-
$t = 1$	-	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	?

Initial effect, $t = 0$

- Obviously A_t increases
- Y_t increases because the economy is more productive (direct effect). C_t increases because of a wealth effect (feel richer, want to consume more).
- N_t increases because working is much more valuable. As stated above,this dominates the wealth effect to increase
- No direct effect on I_t if R_t held constant
- The forces dictating supply for goods (labor) increase drastically compared to Y_t^d, so there is a large gap between output demand and supply. Therefore the interest rate R_t falls to stimulate demand for goods

General equilibrium effect, $t = 0$

- Technology is exogenous (not affected by R_t, so no change.
- C_t is stimulated by interest rate falling (less incentive to save)
- N_t has a small negative effect; we care less about saving/would rather get more utility today so there's a small positive effect on leisure. Y_t has a small negative effect as a result
- I_t increases The opportunity cost on investment is lower. So firms invest more in capital so the return on capital will be equal to the opportunity cost of investing. This is seen explicitly in the intertemporal Euler equation: r_t has a big change, A_t has no change, and N_t has a small change. So K_{t+1} must be increasing a lot for equality to hold. Similarly, the output gap decreases back to 0.

$t = 1$ effect

- C_t is still above steady state because we have more than steady state capital and are "eating our way back" (consumption smoothing)
- N_t is slightly lower (than steady state). Small wealth effect, substitution effect no longer holds.
- Y_t is slightly higher than SS. We are working a little less but have a lot more capital.
- I_t is slightly lower than SS. We are compensating for being above steady state capital,since the shock has already dissipated. R_t begins slowly building back towards steady state (slope of consumption path)

Persistent shock ($1 > \rho > 0$) if tech follows AR(1)

period	A_t	C_t	Y_t	I_t	N_t	R_t
$t = 0$, initial	$\uparrow$	$\uparrow\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
$t = 0$, GE (response to $R_t \uparrow$)	-	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	-
$t = 1$	-	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	?

$t = 0$ effect (mostly focusing on initial)

- Obviously A_t increases
- C_t increases even more than the temporary shock case because the change to the discounted permanent income is greater (stronger wealth effect)
- N_t increases but by less than the temporary since the substitution effect doesn't dominate by as much
- Because $\mathbb{E}_t[\widehat{A}_{t+1}] > 0$, I_t increases
- So the difference between the initial effect of the temporary and persistent shocks is that the degree of changes are different (with the exception of investment). So this time, the demand for goods is experiencing more of an increase than supply (for sufficiently high ρ). So the output gap is in the opposite direction, meaning R_t goes up.
- So the responses to the GE effect are also the opposite. Demand for C_t and I_t should responds negatively, while the small wealth effect now works against leisure (meaning output increases since capital today is a state variable)

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t e^{\epsilon_t} \big(\frac{C_t^{1-\gamma}}{1-\gamma} + \frac{h_t^{1+\eta}}{1+\eta} + \lambda_t [(K_t U_t)^{\alpha} h_t^{1-\alpha} - C_t] \big)$$

We know $K_{t+1} = \big(1 - \frac{U_t^{1+\phi}-1}{1+\phi}\big) K_t$. Therefore

$$(K_{t+1} U_{t+1})^{\alpha} h_{t+1}^{1-\alpha} = \left(\big(1 - \frac{U_t^{1+\phi} - 1}{1+\phi}\big) K_t U_{t+1} \right)^{\alpha} h_{t+1}^{1-\alpha}$$

So then it follows that the F.O.C w.r.t U_t is

$$\lambda_t \frac{Y_t}{U_t} = \mathbb{E}_t[\lambda_{t+1} \beta e^{\epsilon_{t+1} - \epsilon_t} U_t^{\phi} \big(1 - \delta(U_t)\big)^{\alpha-1} \big(K_t U_{t+1}\big)^{\alpha} h_{t+1}^{1-\alpha}] = \mathbb{E}_t[\lambda_{t+1} \beta e^{\epsilon_{t+1} - \epsilon_t} Y_{t+1}] \frac{U_t^{\phi}}{1 - \delta(U_t)}$$

We also know that $\lambda_t = C_t^{-\gamma}$ and $Y_t = C_t$, therefore

$$\frac{Y_t^{1-\gamma}}{U_t^{1+\phi}} (1 - \delta(U_t)) = \mathbb{E}_t[\beta e^{\epsilon_{t+1} - \epsilon_t} Y_{t+1}^{1-\gamma}]$$

Missing: $\frac{1}{U_t^{1+\phi}}(1 - \delta(U_{t+1})) + 1$ (inside the expectation multiplied by $Y_{t+1}^{1-\gamma}$)

8 Supplemental Notes

8.1 Solow

w/ Population Growth Assume there is one household with N_t members. Assume there is a time-invariant rate population growth, so $N_{t+1} = (1+n)N_t$. Now, we have $I_t = sY_t = sF(K_t, N_t)$, where we still normalize the labor supplied per-member of the population to be 1. From the law of motion for capital, we do the "divide by, multiply" trick

$$\frac{K_{t+1}}{N_{t+1}}\frac{N_{t+1}}{N_t} = (1-\delta)\frac{K_t}{N_t} + sF(\frac{K_t}{N_t}, 1) \implies (1+n)k_{t+1} = (1-\delta)k_t + sf(k_t)$$

since $F(\cdot)$ is CRTS. This implies at steady state $(n+\delta)k = sf(k)$. We can also get the golden rule by doing a similar transformation to the resource constraint and then imposing a steady state

$$\frac{C_t}{N_t} + \frac{K_{t+1}}{N_{t+1}}\frac{N_{t+1}}{N_t} - (1-\delta)\frac{K_t}{N_t} = F(\frac{K_t}{N_t}, 1) \implies c = f(k) - (n+\delta)k$$

If we define the steady state variables as functions of s and take a FOC w.r.t s, then we clearly see $f_k(k) = n+\delta$ yielding the golden rule.

(also) w/ Technological Progress We amend the previous discussion to consider a situation where $Y_t = F(K_t, X_tN_t)$ and X_t represents the *labor augmenting* technology (i.e technology that makes each worker more efficient). We assume a technical rate of technological progress and $X_t = (1+x)^t$. So do a similar trick as above,instead normalizing by X_tN_t and define $\widetilde{z} = z/(XN)$ as z per unit of *effective labor*

$$\frac{K_{t+1}}{X_{t+1}N_{t+1}}\frac{X_{t+1}N_{t+1}}{X_tN_t} = (1-\delta)\frac{K_t}{X_tN_t} + sF(\frac{K_t}{X_tN_t}, 1) \implies (1+x)(1+n)\widetilde{k}_{t+1} = (1-\delta)\widetilde{k}_t + sf(\widetilde{k}_t)$$

which yields a slightly less intuitive steady state of $(x+n+nx+\delta)\widetilde{k} = sf(\widetilde{k})$. Again we can do the same to the resource constraint and then impose the steady state

$$\frac{C_t}{N_t} + \frac{K_{t+1}}{X_{t+1}N_{t+1}}\frac{X_{t+1}N_{t+1}}{X_tN_t} - (1-\delta)\frac{K_t}{X_tN_t} = F(\frac{K_t}{X_tN_t}, 1) \implies \widetilde{c} = f(\widetilde{k}) - (n+x+nx+\delta)\widetilde{k}$$

again, defining these variables as a function of s and then taking a FOC w.r.t s, we get that the golden rule solves $f_k(k) = n+x+nx+\delta$

Further Intuition on Dynamics: Some of the content in the Solow Open Econ sections makes more sense once we've gotten to the full neoclassical section after the dynamic programming. For instance, if all of the notes prior to this appendix have been read, we now know that along the locus of points $f(k_t) - \delta k_t$ we have a steady state for capital. We can also provide some more background on monotonistic convergence. intuition on monotonicity. For the proof, we saw that by proving the derivative of the quasi-policy function was bounded between 0 and 1, the FTC showed that k_t was growing (shrinking) if k_0 was below (above) the steady state. But there can be a further appeal made to fixed point theory about why this should follow. Assume $0 < g'(x_t) \leq 1 - \epsilon_t$ for some $\epsilon > 0$. By the Mean Value Theorem,

$$|x_{t+1} - x_t| = |g(x_t) - g(x_{t-1})| \leq (1-\epsilon)|x_t - x_{t-1}| \implies |x_{t+1} - x_t| \leq (1-\epsilon)^t||x_1 - x_0|$$

by induction, where the RHS is converging towards 0.

8.2 Neoclassical Models

Further Intuition on Dynamics: One important thing to note is that we often normalize the rate (e.g. see the rate of population growth in the solow example above) to be 1 + the rate (consider this the "change rate"); this is because when 1 + the rate is multiplied by the current value it yields the new value. In the context of this model, we get the change rate for capital (rate of return + 1) as a ratio of the current LM to tomorrow's LM. This may perhaps make more sense when we look at bonds: the change rate is the inverse of price ($\lambda_t/\lambda_{t+1} = q_t^{-1}$). So when we multiply both sides by the ratio of tomorrow's LM to the current, it should make some intuitive sense that we get 1 on both sides. For capital, this is analogous to the fact that the discount rate is the inverse as the change rate for capital at the steady state. From this, we can perhaps get some more context for the "no arbitrage" property of the model. To expand upon a case that was alluded to in this section, consider $r_t = \rho$ and $k_0 = k^{\text{GR}}$. At the golden rule, the marginal product of capital is δ, meaning the rate of return to capital is 0. Thus, there is an incentive to sell off capital and buy bonds because they obviously have a higher rate of return. There can be some further background on this issue by considering a general discussion of income and substitution effects, shown below. Another relevant issue to consider in the small open economy is the issue of "jumping straight to a steady state". This may not make sense, considering some of the discussion in the closed neoclassical section. But in this setup, it was assumed that the interest rate would be constant, implying that consumption is also constant. Since there can't be a difference in the rates of return, the capital level must just straight to k^{NM}. If $k_0 < k^{\text{NM}}$, the bond market allows for you to borrow the exact right amount (consider negative bond holdings) to be able to consume at a constant rate. If $k_0 > k^{\text{NM}}$, the household will buy bonds and sell capital to be able to consume at a higher permanent level, with the ROW paying them back (this scenario was discussed more explicitly in the earlier notes). In the classic, closed economy neoclassical model, there is no ROW, so in order to jump, this would have to be through a very high savings rate, which would eat at a proportionally much higher chunk of consumption compared to what was seen with the sacrifice made in the open economy setup. Further, if we assume one is willing to absorb all that "pain" today, that suggests that the interest rate must at least be higher than the rate of time preference. Qualitatively thinking of this as a relatively high rate in and of itself, this means people are discouraged from borrowing (in a mathematical sense with respect to our model, a high interest rate implies a low price for bonds). But this also yields a contradiction for the desire to jump straight to a higher level of capital: a high interest rate implies a high marginal product of capital tomorrow (whereas a jump should imply a lower marginal product). In more simple terms, the fact that a bond market in a closed economy bust clear at every period and has endogenous prices means that to make a jump, there must be drastic substitution away from capital, which doesn't really make sense given the dynamics of the model.

Substitution and Income Effect Consider an application to the "one period" Robinson Crusoe model (which has leisure); this makes the notation simpler although the setup and results are virtually identical temporal models. Suppose the level of technology increases ($A' > A$). Then we can consider the *substitution effect* what could happen to (h, c) if we held marginal utility constant (i.e. stayed on the same indifference curve). Since obviously the curve $A'F(k, h)$ would yield an upward shift relative to $AF(k, h)$, we can consider doing this by considering a function $A'F(k, h) + B$, where $B < 0$ can be thought of as forcefully hindering the level output output ("production technology") so we can stay on the same indifference curve. If we assume preferences over consumption and leisure are quasi-concave[5], the substitution effect yields increases in both h and c. Here is a formal proof of claim. Consider $c = B + AF(k, h)$ and $AF_h(k, h) = U_\ell(\cdot)/U_h(\cdot)$. Total differentiation of the utility function yields $dU = U_c dc - U_\ell dh = 0$, which through substitution in the second equation yields $dAF_h = dc/dh$. This equality implies that the derivatives of c and h with respect to A will be signed the same in this instance. By substituting this into a total differentiation of the first equation, we get $dB = -F \cdot dA$, meaning we get the desired substitution effect property that utility does not change with an increase in A. Finally, totally differentiating some of these results and substituting in some of the previous results,

$$[(-U_\ell^2 U_{cc} + 2U_\ell U_c U_{c\ell} - U_c^2 U_{\ell\ell}]\frac{1}{U_c^2 U_\ell} - AF_{hh}\frac{U_c}{U_\ell}] \cdot dc = F_h \cdot dA$$

Working through each term on the LHS, revisiting the assumptions on the relevant functions will show they are all positive (in particular $U_{c\ell} > 0$ because of quasiconcavity). So $dc/da > 0$, meaning consumption and consequently labor rises with an increase in the level of technology (this is, again, under the imposition of holding utility constant through the B term). The result should also follow intuitively, because a higher level technology implies an increase in marginal product of labor, which means that leisure is more expensive.

[5]for our application, this can be thought of simply as both monotonistic and convex, with an implication that indifference curves get steeper as h increases. A more formal applied way of framing this is if we have two bundles of consumption and labor (WLOG consider $c_1 > c_2$ and $h_1 > h_2$) such that utility is equal, then $\exists (h, c) \in \big([h_1, h_2], [c_1, c_2]\big)$ yielding higher utility

So what you can extrapolate is that in this instance (where we stay on the same indifference curve) the benefits to utility by consuming more are exactly offset by the disutility from working more. However, this is an offsetting process that we imposed ourselves to stay on the same indifference curve. We could in fact jump to a different indifference curve (i.e. achieve higher utility) by simply letting the curve shift outward, recognized as the *income effect*. Here, we see that a given time interval of work is more effective, so one could work less and eat the same amount. This is why the effect on labor/hours is ambiguous: there is a competing interest between leisure becoming more expensive but at the same time not needing to work as hard to get a certain amount of consumption. The precise nature of the shift rectifies this ambiguity. See the table and graph for more intuition.

8.3 Stochastic

AR(p) as MA(∞) This is directly from stack exchange[6]. For an AR(p) series

$$\underbrace{(1-\rho_1 L-\rho_2 L^2-\cdots\rho_p L^p)}_{\Phi(L)}x_t=u_t,$$

where the polynomial $\Phi(z)$ has roots strictly outside the unit circle, the ψ-weights in the causal MA(∞) representation $x_t=\sum_{i=0}^{\infty}\psi_i u_{t-i}$ are the solutions to the difference equations

$$\begin{aligned}\psi_0&=1\\ \psi_1-\rho_1\psi_0&=0\\ \psi_2-\rho_1\psi_1-\rho_2\psi_0&=0\\ &\vdots\\ \psi_{p-1}-\rho_1\psi_{p-2}-\cdots\rho_{p-1}\psi_0&=0\\ \psi_t-\rho_1\psi_{t-1}-\rho_2\psi_{t-2}\cdots\rho_p\psi_{t-p}&=0,\ \ \forall t\geq p.\end{aligned}$$

The system can be solved like any linear homogeneous system of difference equations. The solution $\{\psi_i\}$ is a linear combination of terms of the form r^{-t} where r is a root of the AR polynomial Φ. (The cases of repeated or complex roots are ignored for simplicity. Same result holds.) The causality assumption ensures that $x_t=\sum_{i=0}^{\infty}\psi_{t-i}u_i$ converges, as a random variable. (In the AR(1) case, the causality condition is $|\rho|< 1$.)

When the AR polynomial has roots possibly inside, but not on, the unit circle. The MA(∞) representation still exists but is not causal in general, i.e. it can be two-sided

$$x_t=\sum_{-\infty}^{\infty}\psi_i u_{t-i}.$$

For example, in the AR(1) case with $|\rho|>1$, then non-causal MA representation is a forward-looking solution

$$x_t=\sum_{-\infty<i\leq-1}(\frac{-1}{\rho})^{-i}u_{t-i}.$$

8.4 MISC

Taylor Series Expansion These largely follow Eric Sims' notes.
Taylor's theorem tells us the following

$$f(x_t)=f(x)+f'(x)(x_t-x)+\frac{f^{(2)}(x)}{2!}(x_t-x)^2+\frac{f^{(3)}(x)}{3!}(x_t-x)^3+\ldots$$

[6] https://stats.stackexchange.com/questions/455679/how-to-recursively-express-an-arp-process/455852#455852

where the expansion is considered "at x", usually meaning the steady state in the context of Macro. For smooth functions, the magnitude of the terms dissipates pretty quickly as n (order) grows. Therefore, we usually (especially throughout the notes) just consider

$$f(x_t) = f(x) + f'(x)(x_t - x) \ \underline{\textbf{and}}\ f(x_t, y_t) = f(x, y) + f_x(x, y)(x_t - x) + f_y(x, y)(y_t - y)$$

where equality is imposed <u>but it's really an approximation</u>. We also have a useful trick by *log-linearizaiton*. The usual definition for a linearized variable is $\frac{x_t - x}{x} = \widehat{x}_t$. We can use a Taylor expansion trick, since we assume $\widehat{x}_t$ to be small in magnitude. We have from a first order Taylor expansion about $\widehat{x}_t = 0$ to be

$$\ln(1 + \widehat{x}_t) \approx \ln(1) + \frac{d}{d\widehat{x}_t} \ln(1 + \widehat{x}_t)\bigg|_{\widehat{x}_t = 0} \times (\widehat{x}_t - 0) = \frac{1}{1 + \widehat{x}_t}\bigg|_{\widehat{x}_t = 0} \times \widehat{x}_t = \widehat{x}_t$$

So now consider the following property, which is extremely useful

$$\widehat{x}_t \approx \ln(1 + \widehat{x}_t) = \ln(1 + \frac{x_t - x}{x}) = \ln(\frac{x_t}{x}) = \ln(x_t) - \ln(x)$$

To see the utility in this simplification, consider the following

$$y_t = x_t z_t \implies \widehat{y}_t \approx \ln(y_t) - \ln(y) = \big(\ln(x_t) + \ln(z_t)\big) - \big(\ln(x) - \ln(z)\big) \approx \widehat{x}_t + \widehat{z}_t$$

This gives us the first in several rules/properties for this process

- $y_t = x_t z_t \implies \widehat{y}_t = \widehat{x}_t + \widehat{z}_t$
- $y_t = x_t^\alpha \implies \widehat{y}_t = \alpha \widehat{x}_t$
- $y_t = f(x_t) \implies \widehat{y}_t = \big[\frac{f'(x)}{f(x)} x\big]\widehat{x}_t$
- $y_t = x_t + z_t \implies y\widehat{y}_t = x\widehat{x}_t + z\widehat{z}_t$

To get some intuition for the third rule, consider $f(x_t) = \frac{g(x_t)}{h(x_t)} \implies \ln(f(x_t)) = \ln(g(x_t)) - \ln(h(x_t))$. So

$$\ln(f(x_t)) = \ln(f(x)) + \frac{f'(x)}{f(x)}(x_t - x), \ln(g(x_t)) = \ln(g(x)) + \frac{g'(x)}{g(x)}(x_t - x), \text{and} \ln(h(x_t)) = \ln(h(x)) + \frac{h'(x)}{h(x)}(x_t - x)$$

by a Taylor series expansion with equality imposed. so if we recognize that $\ln(f(x)) = \ln(g(x)) - \ln(h(x))$, then imposing these equalities derived from the Taylor expansion into the original relation with logs you get

$$\frac{f'(x)}{f(x)}(x_t - x) = \frac{g'(x)}{g(x)}(x_t - x) - \frac{h'(x)}{h(x)}(x_t - x) \implies \frac{xf'(x)}{f(x)}\widehat{x}_t = \frac{xg'(x)}{g(x)}\widehat{x}_t - \frac{xh'(x)}{h(x)}\widehat{x}_t$$

A more simple way to see this is also

$$\widehat{y}_t \approx \ln(y_t) - \ln(y) = \ln(f(x_t)) - \ln(f(x)) \approx \big(\ln(f(x)) + \frac{f'(x)}{f(x)}(x_t - x)\big) - \ln(f(x)) = \frac{f'(x)}{f(x)}(x_t - x) = \frac{xf'(x)}{f(x)}\widehat{x}_t$$

<u>*Example:*</u> Consider $k_{t+1} = (1 - \delta)k_t + sA_t k_t^\alpha$. This means

$$\begin{aligned} \widehat{k}_{t+1} &= \frac{(1-\delta)k}{k}\widehat{(1-\delta)k_t} + \frac{sAk^\alpha}{k}\widehat{sA_t k_t^\alpha} && \textbf{Rule 4} \\ &= (1-\delta)[\widehat{(1-\delta)} + \widehat{k}_t] + sAk^{\alpha-1}(\widehat{s} + \widehat{A}_t + \widehat{k}_t^\alpha) && \textbf{Rule 1} \\ &= (1-\delta)\widehat{k}_t + \delta Ak^{\alpha-1}(\widehat{A}_t + \alpha\widehat{k}_t) && \textbf{Rule 2} \end{aligned}$$

where we initially treat the whole term as one linearized variable and recognized that a linearized constant is simply 0. Notice this is equivalent to the earlier result in the notes when a Taylor expansion was used.

Dictionary

Math
f_{xx} is the second derivative of f wrt x
$\exists$ - there exists
$\forall$ - for all
$\therefore$ - therefore

Terms
BC - Budget Constraint
CRTS - Constant Returns to Scale: $\alpha F(x,y) = F(\alpha x, \alpha y)$
FOC - First Order Condition
GR - Golden Rule
HH - Household
LHS/RHS - Left/Right Hand Side
LM - Lagrange Multiplier
s.t - such that
wrt - with respect to
"zero coupon" bond market - Pays out in its entirety at maturity, which for our cases is the next period