Real Analysis I

Acknowledgements

These notes follow closely from Chapters 1-7 Rudin's *Principles.* These are notes compiled from classes I have taken from Dr. Logan Stokols (Duke), Dr.David Cruz-Uribe, and Dr.Tim Ferguson (both UA). In particular, the topics not covered in Rudin and the homework problems at the end are from Dr. Stokols' class. I would also like to emphasize Dr. Cruz-Uribe's notes as being particularly helpful and comprehensive. All these instructors were incredible and personable in their approach/interactions; I can't thank them enough. As far as the format of this textbook, there is little extraneous discussion or commentary. These are mostly definitions and results stated in a linear fashion. There is a lot of shorthand used, some of which is defined after the notes. This was crafted with only myself in mind but may be helpful to others. Also: I corrected the HW problems I was marked off for but cannot guarantee there are no mistakes

1 Real and Complex number systems

lots of information skipped for this chapter

A **totally ordered** set is a set S with a *total order* $<,>,\leq,\geq$ s.t $\forall\, x,y,z \in S$ either $x<y$ or $x>y$ or $x=y$ and (transitivity) if $x<y, y<z \implies x<z$.
An **ordered field** is a *field* F (satisfies usual addition and multiplication axioms, see Rudin) that is an ordered set and $x,y,z \in F, y<z \implies x+y<x+z$ and $x,y \in F, x,y>0 \implies xy>0$.

Theorem (Cauchy-Schwarz Inequality) For any ordered field F, $\forall\, x,y \in F\;\; xy \leq .5(x^2+y^2)$

An ordered set S is **(Dedekind) complete**, or has the *least upper bound property*, if $\forall E \subseteq S$ bounded above and nonempty $\exists \alpha \in S$ s.t $\alpha = \sup E$.

Theorem For $x \in \mathbb{R}$ and $n \in \mathbb{N}$ $\exists\, y \in \mathbb{R}$ s.t $y = x^n$
Proof: This is a proof for $\exists y \in \mathbb{R}^+$ s.t $y^3 = 3$, which can be adapted for the general case by using different constraints (i.e. w.r.t x and n instead of 3). Define the set $E = \{x \in \mathbb{R}^+ | x^3 < 3\}$. We know $\sup E$ exists by the (Dedekind) completeness of $\mathbb{R}$. We want to show that $(\sup E)^3 = 3$.
Case 1 Define arbitrary $\alpha > 1$ s.t $\alpha^3 < 3$. We will show α isn't an upper bound. Let $\delta = 3 - \alpha^3$, implying $\delta \in (0,1)$. Fix $\epsilon > 0$ s.t $\epsilon < \delta/(9\alpha^2) < \delta/9$. Note that for $n \in \mathbb{N}\backslash\{1\}$ $\alpha^n > \alpha$ and $\epsilon^n < \epsilon$. Then

$$(\alpha+\epsilon)^3 = \alpha^3 + \epsilon^3 + 3(\alpha\epsilon^2 + \alpha^2\epsilon) < \alpha^3 + \epsilon(1+6\alpha^2) < \alpha^3 + (7\delta)/9 < \alpha^3 + \delta = 3$$

Therefore, $\alpha + \epsilon$ is not an upper bound of E, so neither is α
Case 2 Now define $\alpha \in (1,2)$ s.t $\alpha^3 > 3$. We will show α is not the least upper bound of E. Let $\delta = \alpha^3 - 3$, so $\delta \in (0,1)$. Fix $\epsilon > 0$ s.t $\epsilon < \delta/(6\alpha^2) < \delta/6$. Since $\epsilon^3 > 0$, $-\epsilon^2 > -\epsilon$, and $-\alpha > -\alpha^2$

$$(\alpha-\epsilon)^3 = \alpha^3 + \epsilon^3 - 3(\alpha\epsilon^2 + \alpha^2\epsilon) > \alpha^3 - 6\epsilon\alpha^2 > \alpha^3 - \delta = 3$$

Therefore, α can't be the least upper bound because $\alpha - \epsilon$ is an upper bound
As mentioned, we know $\sup E$ exists. Further, $\sup E \in \mathbb{R}^+$ from the cases above. Define $y = \sup E$. In both cases, we defined α arbitrarily, meaning that we can make the general statement that for $u \in \mathbb{R}$, if $u^3 < 3$ or $u^3 > 3$, then $u \neq \sup E$. By contraposition, $y^3 = (\sup E)^3 = 3$ ■.

Theorem (Density of $\mathbb{Q}$) $\forall x,y \in \mathbb{R}\; x>y \implies \exists\, p \in \mathbb{Q}$ s.t $x>p>y$

Theorem (Archemedian Property) $x,y \in \mathbb{R}^+ \implies \exists\, n \in \mathbb{N}$ s.t $nx > y$. Also, $\exists\, m \in \mathbb{Q}$ s.t $m > x > m^{-1}$.

Dot Product $x \cdot y = \sum_{i=1}^n x_i y_i$
For $n \in \mathbb{N}$, the n-dimensional **Euclidean Space** is $\mathbb{R}^n$ (n-tuple) with dot product and norm $||x|| = \sqrt{x \cdot x}$

Theorem (Cauchy-Schwarz) $x,y \in \mathbb{R}^n \implies x \cdot y \leq ||x|| \cdot ||y||$
Proof: $0 \leq ||x+ty||^2 = ||x+ty||^2 = x \cdot x + 2t(x \cdot y) + t^2(y \cdot y) = ||x||^2 + (2t)x \cdot y + (t||y||)^2$
The last term is a non-negative quadratic ($\forall t$), so all $D \leq 0$ (discriminants, e.g. $4(x \cdot y)^2 - 4||x||^2||y||^2$) since a quadratic $p(t) \geq 0\; \forall t$ intersects the "real 0" y-axis once or never, so it has one or less real roots

Triangle Inequality $x,y \in \mathbb{R}^n \implies ||x+y|| \leq ||x|| + ||y||$
Proof: $||x+y||^2 = ||x||^2 x \cdot y + ||y||^2 \leq ||x||^2 + 2||x|| \cdot ||y|| + ||y||^2 = (||x|| + ||y||)^2$

2 Basic Topology

Let $f : A \to B$. If for each $y \in B, f^{-1}(y)$ consists of at most one element of A, the f is a *one-to-one* mapping. A is finite if $\exists$ a 1-1 mapping onto a finite set J_n. A is **countable** if $\exists$ a 1-1 mapping onto $\mathbb{N}$.

Theorem Let $\{E_n\}$ be a sequence of countable sets. Then $S = \cup_{n\in\mathbb{N}} E_n$ is countable

Proof: Let all E_n be countably infinite. Consider each E_n on a row, such that set i's jth element is in the i, jth position in an infinite array. However, we can also consider that combining i and j yields a natural number (e.g. 1010 w/ $i = j = 10$) and this integer will be unique for every $i, j \in \mathbb{N}$. Further, each combination will yield an integer greater than 11. Therefore, there some subset of $\mathbb{N}$, call it T, such that there is a 1-1 correspondence from S to T. Because S is a union of countably infinite sets, it must also be countably infinite.

A **metric space** is a set of points X, together with a *metric* $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ s.t i) (pos. definite) $d(x, y) = 0$ iff $x = y$ ii) (symmetry) $d(x, y) = d(y, x)$ iii) triangle inequality w.r.t the metric holds

For a metric space X and $r \in \mathbb{R}^+$, the (open) **ball** $B_r(x)$ is $\{y \in X | d(x, y) < r\}$

$E \subseteq X$ is **open** if $\forall x \in E \;\exists r > 0$ s.t $B_r(x) \subseteq E$

Theorem Open balls are open

Proof: X metrix space, $x \in X$, and $r > 0$. Let $y \in B_r(x)$, $\delta = d(x, y) < r$, and some $\rho \in (0, r - \delta)$.
For any $z \in B_\rho(y)$, $d(x, z) \leq d(x, y) + d(x, z) < \delta + r - \delta = r$. So $B_\rho(y) \subseteq B_r(x)$.

Theorem If $E \subseteq X$ open, $\exists$ family of open balls $\{B_\alpha\}_{\alpha\in I}$ s.t $E - \cup_{\alpha\in I} B_\alpha$

Proof: $\forall x \in E$, $\exists r > 0$ s.t $B_r(x) \subseteq E$. Denote this ball as B_x.
Each $B_x \subset E$, so $\cup_{x\in E} B_x \subset E$. $B_y \ni y \in E$, so $E \subset \cup_{x\in E} B_x$

$E \subseteq X$ is *bounded* if $\exists r > 0, x \in X$ s.t $E \subseteq B_r(x)$

For metric space X and $E \subseteq X$ (*compliment* $E^c = X \backslash E$),
- E°, the **interior** of E, is the set of all $x \in X$ s.t $B_\epsilon(x) \subseteq E$ for some $\epsilon > 0$
- ∂E, the **boundary** of E, is the set of all $x \in X$ s.t $\forall \epsilon > 0$, $B_\epsilon(x) \cap E^c, B_\epsilon \cap E \neq \emptyset$
- E', set of all **limit points** of E, s.t $x \in E'$ iff $\forall r > 0$, $\exists y \in B_r(x)$ where $y \neq x, y \in E$
- $\overline{E}$, the **closure** of E, is the set of all $x \in X$ s.t $\forall r > 0$, $B_r(x) \cap E \neq \emptyset$
- x is **isolated** if $x \in E$ but $x \notin E'$
- E is **dense** in X if $x \in X \implies x \in E'$ and/or $x \in E$

Set Theory Propositions (resulting from above definitions) For any $E \subseteq X$

a) E is open iff $E = E^\circ$
b) $E', \partial E, E \subseteq \overline{E}$
c) $p \in E' \implies B_r(p)$ contains infinitely many points of $E (\forall r > 0)$. E finite $\implies E' = \emptyset$.
d) $\partial E = \overline{E} \backslash E^\circ$
e) $\overline{E} = E \cup E' = E \cup \partial E$

Proof: First note prop b. Then, for $x \in \overline{E}, x \notin E \implies$ every x-ball contains point of E, can't be x itself, so $x \in E'$. Also, every x-ball contains $y \in E^c$, so $x \in \partial E$

$E \subseteq X$ is **closed** if any of the following hold (iff)

a) $E = \overline{E}$
b) $E' \subseteq E$ (Rudin: every limit point is a point of E)
c) $\partial E \subseteq E$
d) E^c is open

Proof: ($a \implies d$) If $x \in E^c$, then $x \notin \overline{E}$, so $\exists r > 0$ s.t $B_r(x) \cap E = \emptyset$ (def of closure) or $B_r(x) \subseteq E^c$
Proof: ($d \implies a$) If $x \in E^c$, $\exists r > 0$ s.t $B_r(x) \subseteq E^c$ or $B_r(x) \cap E = \emptyset$, so $x \notin \overline{E}$ (and $x \notin E$)

Theorem Let $\{G_\alpha\}$ be a collection of open sets. $\cup_\alpha G_\alpha$ is open and finite intersections are open. If $\{G_\alpha\}$ is a collection of closed sets, $\cap_\alpha G_\alpha$ is closed and finite unions are closed

Proof: (open) $x \in \cup_\alpha G_\alpha \implies x \in G_a$ (some a). G_a open $\implies \exists r > 0$ s.t $B_r(x) \subseteq G_a \subseteq \cup_\alpha G_\alpha$.
Let $B \subseteq \mathbb{N}$ be finite and consider $\{G_\beta\}_{\beta\in B}$. $x \in \cap_\beta G_\beta \implies x \in G_b \ \forall b \in B$, so $\exists r_b > 0$ s.t $B_{r_b}(x) \subseteq G_b$. Let r be the minimum of all such r. Then $B_r(x) \subseteq \cap_\beta G_\beta$

Theorem: $\overline{E}$ is closed. If $E \subseteq F$ with F closed, then $\overline{E} \subset F$.

Proof: $x \notin \overline{E} \implies \exists B_r(x)$ s.t $B_r(x) \cap E = \emptyset$. Since this is true $\forall x \notin \overline{E}$, $\overline{E}^c$ is open, so $\overline{E}$ is closed.
F closed $\implies F' \subset F \implies E' \subset F$ if $E \subset F$. So $E \cup E' = \overline{E} \subset F$
Intersection of all closed sets: from b, $\overline{E} \subset \cap F$. But $\overline{E}$ closed and $E \subset \overline{E}$ so taking $F = \overline{E}$ gives $\cap F \subset \overline{E}$

A, B are **separated** if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. $E \subseteq X$ **disconnected** if $\exists A, B \subseteq X$ nonempty s.t they are separated with $E = A \cup B$. Say A, B *separate* E. A set which isn't disconnected is **connected**

If $A \subseteq X$ closed & open, and $\emptyset \neq A \neq X$, then X disconnected

Proof $\overline{A} = A$, $\overline{A}^c = A^c$ (open $\implies$ compliment closed), and $A \cap A^c = \emptyset$

If A, B separated, E connected, then $E \subseteq A \cup B \implies E \subseteq A$ or $E \subseteq B$

Proof: $E \cap A$ and $E \cap B$ separated because $\overline{E \cap A} \subseteq A$. If both nonempty, E disconnected

If E, F connected and $E \cap F \neq \emptyset$, then $E \cup F$ connected

If E connected can draw paths; path connected $\implies$ connected, not iff (comb and flea)

X metric space, a connected component A of X is a *maximal connected subset*: A connected if $A \subseteq B \subseteq X$ and B connected then $A = B$

Connected components of X partition X: pairwise disjoint, union is X, and are closed. If X has finitely many connected components, they are open.

Sketch: $\forall x \in X$, define A_x as the union of all sets E s.t $x \in X$ and E is connected. Clearly connect all of X, can show A_x connected, and $\forall x, y$, $A_x = A_y$ or $A_x \cap A_y = \emptyset$. Ex: c.c of $\mathbb{Q}$ are singletons $\{q\}(\ \forall q \in \mathbb{Q})$

Every continuous function on $[0, 1]$ is bounded.

An **open cover** of $E \subseteq X$ is a a collection $\{G_\alpha\}_{\alpha\in A}$ of open sets G_α of open sets $G_\alpha \subseteq X$, $E \subseteq \cup_{\alpha\in A} G_\alpha$. A **subcover** is a collection $\{G_\beta\}_{\beta\in B}$ s.t $B \subseteq A$ and it covers E. Cover is finite if A is finite.

$E \subseteq X$ is **compact** if every open cover has a finite subcover.
$\{0, 1, .5, .33, .25...\}$ compact. Take 0 off it's not.

Theorem Compact sets are closed.

Proof: If $x \notin K$, then $\forall y \in K$, let $r_y = d(x, y)/3 > 0$. By triangle inequality (ensured by diving through by 3), $B_{r_y}(x) \cap B_{r_y}(y) = \emptyset$. Infinity problem since boundedness not assumed, so now exploit compactness: $B_{r_z}(y)\}_{y\in K}$ is open cover; let $\{y_i\}_{n=i}^N$ centers of a finite subcover. $V = \cap_{i\in[1,N]} B_{r_{y_i}}(x) \subset K^c$, so K^c open

Theorem Any compact set is bounded.

Proof: Choose any $x \in X$. Then $\{B_n(x)\}_{n\in\mathbb{N}^+}$ is open cover of K. Finite subset of $\mathbb{N}^+$ will suffice, let N be the largest. $K \subseteq B_N(x)$

Theorem Let $K \subseteq X$ compact, $E \subseteq K$ infinite. Then E has limit point in K

Proof: Suppose not. Then every $x \in K$ has B_x s.t $B_x \cap E$ is at most one point (x itself if $x \in K$). $\{B_x\}_{x\in K}$, open cover of K, finite subcover contain at most finitely many (contradiction to infinite assumption).

$E \subset Y$ is **open relative** to Y if for each $p \in E$ $\exists r > 0$ s.t $q \in E \implies d(p,q) < r$. A set is **closed relative** to Y iff its the compliment of a set open relative to Y.

Define an *equivalence set* $[B_r(x_0)]_A$ by $\{x \in A | D(x, x_0) < r\}$

Theorem $E \subset Y \subset X$ is open relative to Y iff (for some $G \subset X$ open) $E = Y \cap G$. E closed rel to y, then E is closed rel to X iff $\overline{E} \subseteq Y$. $E \subseteq Y$ closed rel to $Y \subseteq X$ iff $\exists F \subset X$ closed s.t $E = F \cap Y$.

Proof: (2nd result) $\Longleftarrow$ Know $E = F \cap Y$ for some $F \subseteq X$ closed. Then $\overline{E} \subseteq F$ because F closed and $\overline{E} \subseteq Y$. $E \subset \overline{E} \subset F \cap Y$, so $E = \overline{E}$

Ex: $\sqrt{2} \in \overline{\mathbb{Q}} \subseteq \mathbb{R}$, so $\{x \in \mathbb{Q} | x \in [1,2]\}$, closed in $\mathbb{Q}$, not in $\mathbb{R}$

Theorem If $K \subseteq Y \subseteq X$, then K compact relative to Y iff K compact relative to X

Proof Let $\{V_\alpha\}$ be an open cover relative to Y, and $U_\alpha = V_\alpha \cap Y$. Then $\{U_\alpha\}_{\alpha \in A}$ open cover relative to Y has finite subcover for $B \subseteq A$ finite. $K \subseteq \cup_B U_\beta \subseteq \cup_B V_\beta$ (finite subcover)

Theorem $E \subseteq K$ closed, K compact, then E is compact.

Proof: Let $\{G_\alpha\}$ be an open cover of E. Then union of the open cover and E^c an open cover for K, and $\exists B \subseteq A$ finite that creates a finite cover for both K and therefore E

Finite intersection property (FIP) $\{E_\alpha\}_{\alpha \in A}$ s.t $\cup E_\alpha \subset X$ has FIP if $\forall B \subseteq A$ finite, $\cap_B E_\alpha \neq \emptyset$

Theorem K compact iff $\forall$ families of $\{E_\alpha\}_{a \in A}$ of closed sets with FIP $\cap E_a$ nonempty

Proof: (sketch) same as definition as compact, with *not* everywhere (contrapositive). Take $G_\alpha = K \cap E_\alpha$. Then $\{G_\alpha\}$ open cover iff $K \subseteq \cup_A G_\alpha$ iff $\cap E_\alpha \subseteq \emptyset$ *(FIP means no finite subcover - always be something outside)*

If $\{K_\alpha\}$ family of compact subsets of X with FIP, then $\cap K_\alpha \neq \emptyset$. (Idea: if X compact done. If $\cup K_\alpha$ compact, done. Since care about $\cap K_\alpha$ "all action" happens in each/all K_α)

Proof: Fix any $K_0 \in \{K_\alpha\}_{\alpha \in A}$. Set $E_\alpha = K_\alpha \cap K_0$. $\{E_\alpha\}$ closed subsets of K_0 compact. For $B \subseteq A$, $\cap_B E_\alpha = K_0 \cap (\cap_B K_\alpha)$. Nonempty for B finiifte therefore non-empty for $B = A$.

Theorem Define real numbers $a < b$ then $[a,b]$ compact.

Proof: Suppose $\{G_\alpha\}$ open cover of $[a,b]$ with no finite subcover. So at least $[a,c]$ or $[c,b]$ must have no finite subcover. Define iteratively $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, so $|b_n - a_n| \leq 2^{-n}|b-a|$. Since $a_n < b$, $x = \sup\{a_n | n \in \mathbb{N}\}$ exists. Notice $a_n \leq x \leq b_n$. Also, any $b_n \in B_{2^{1-n}|b-a|}(x)$. Some G_α contains x, so there exists $r > 0$, $B_r(x) \subseteq G_\alpha$. Take n s.t $2^{1-n}|b-a| < r$ and $[a_n, b_n] \subseteq G_\alpha$.

Any bounded closed subset of $\mathbb{R}$ is compact.

Heine-Borel any closed and bounded subset of $\mathbb{R}^k$ is compact.

(weirstrauss corallary). Any bounded infinite $E \subseteq \mathbb{R}^n$ has limit point.

Theorem If $E \subseteq \mathbb{R}^n$ and every infinite subset of E has limit point, then E compact

Proof: For each $k \in \mathbb{N}$, if $E \cap \{x | x \in (k, k+1)\}$ is nonempty, choose element, call it x_k. Then the collection of these has no limit point (all isolated since we're choosing for each natural number), must be finite (by the Archemedian property since $E \subseteq \mathbb{R}^n$), so E bounded. So let $x \in E'$. For $k \in \mathbb{N}$, choose $x_k \in B_{\frac{1}{k}}(x) \cap E$. Then $\{x_k\}$ infinite for limit $y \in E$. Assume $x \neq y$, let $0 < \epsilon = d(x,y)/3$.

$$d(y, x_k) \geq d(x,y) - d(x, x_k) = 3\epsilon - d(x, x_k) > 3\epsilon - 1/k$$

If $k > 1/\epsilon$, $x_k \notin B_\epsilon(y)$. So $B_\epsilon(y) \cap \{x_k\}$ finite

3 Sequence and Series

Sequence (p_n) is a map $f : \mathbb{N} \to X$, denoted by $f(n) = p_n$. *Range* of sequence is $\{p_n | n \in \mathbb{N}\}$

p_n **converges** to p $(p_n \to p)$ if $\forall \epsilon > 0\ \exists N$ s.t $n > N \implies d(p_n, p) < \epsilon$
Difference between limit point and range: $(.5, -3, 1/3, -3, .25, \dots)$ 0 is limit point of range, but not (p_n)

Formally, we say a sequence is *divergent* if the sequence is unbounded or if there is no p that satisfies the above definition (i.e. we do not consider $\pm\infty$ a possible limit point)

$x_n \to x$ iff every ball of x contains all but finitely many x_n

Theorem $E \subseteq X$ is closed iff $\forall$ sequences (x_n) in E $x_n \to x \implies x \in E$
Proof: $x_n \to x$ and $x \in E \implies \exists x_m \in B_r(x)$ (some $m \in \mathbb{N}$ and $\forall r > 0$), so $x \in \overline{E} \implies \forall n \in \mathbb{N}$ choose $x_i \in B_{1/n}(x_n)$, $x_n \to x$ by archemdian property (ball shrinks to "0 radius"). If closed E contains all its lps

Bc of $> N$ definition, infinitely many points clustered around limit. $x \in E$ isolated $\implies x \notin E'$.

Theorem Limit of sequence is unique
Proof: $x_n \to p$ and $x_n \to p \implies \forall \epsilon > 0\ \exists M, N \in \mathbb{N}$ s.t $d(x_n, p) < .5\epsilon$ and $d(x_n, q) < .5\epsilon$. Then by triangle inequality $\forall n > \max\{M, N\}$ $d(p, q) \leq d(p, x_n) + d(q, x_n) = \epsilon$. (use positive definite of metric).

A sequence is *bounded* if its range is bounded.

Theorem Any convergent sequence is bounded
Proof: $x_n \to x \implies \exists N$ s.t $d(x_n, x) < 1 (\forall n > N)$. Then the sequence of distances between x and x_i $i \in [1, N]$ is finite. Let R be the max (well-defined since convergence in $\mathbb{R}$ implicit)

If a_n, b_n sequences and $a_n \geq b_n$ for all but finitely many $n \in \mathbb{N}$, or if $(b_n) = a_N, a_{N+1}, \dots$, then either a_n, b_n both divergent or their limits are equal

$x_n \to x$. If $n > N \implies d(x_n, x) < \epsilon$, then $m > N \implies d(x_n, x_m) < 2\epsilon$

A sequence is **cauchy** if $\forall \epsilon > 0\ \exists N$ s.t $m, n > N \implies (d_n, d_m) < \epsilon$ (convergent $\implies$ cauchy)

A space (X, d) is *(Cauchy) complete* if every cauchy sequence in X is convergent in X

Theorem Compact sets are complete 6
Proof: Let (x_n) be cauchy in K compact. If $\exists y$ s.t $x_n \leq y$ for infinitely many $n \in \mathbb{N}$, then $x_n \to y$. Assume WLOG no x_n repeat infinitely often. Let $E_N = \{x_n | n > N\}$. Each E_N infinite and $\overline{E}_N \subseteq K$ is compact, has FIP, so $\exists x \in \cap_N \overline{E_N}$. Fix $\epsilon > 0$. Let M s.t $n, m > M \implies d(x_n, x_m) < .5\epsilon$. Then $x \in \overline{E}_M$, so $B_{.5\epsilon}(x)$ contains an element of the sequence. Then the distance between this point and x will be $< \epsilon$

If $X \subseteq Y$ metric space, Y complete and $\overline{X} = Y$, then call (Y, d) the *cauchy completion* of (X, d).

Theorem Every space has a cauchy completion
Proof: (sketch) Points of Y will be equivalence classes of Cauchy sequences (i.e. if we combine sequences, resulting sequence is Cauchy) in X. Define the LIM X_n the formal limit of every Cauchy (X_n) in X. If $x_n \to x$, LIM X_n represents x. Define a metric

$$\overline{d}(LIMa_n, LIMb_n) = \lim d(a_n, b_n)$$

The limit is cauchy in $\mathbb{R}$, so the $\overline{d}$ term exists. Set $LIMa_n = LIMb_n$ if $\overline{d} = 0$. Set $Y = \{LIMx_n | (x_n) \text{ Cauchy sequence} in X\}$. For $x \in X$, identify $x \sim LIM(x)$, so $X \subseteq Y$. Must show $\overline{d}$ is a metric, $d(x, y) = \overline{d}(LIM(x), LIM(y)$, X dense in Y, $(Y, \overline{d})$ is cauchy complete.

x_n sequence in (X,d) and n_k sequence in $\mathbb{N}$ s.t $n_0 < n_1 < \ldots$. Then $(x_{n,k})_{k\in\mathbb{N}}$ is a subsequence of (x_n). If $x_{n,k} \xrightarrow{k} y$, call y **subsequential** limit of x_n.
Alternative notation: $A \subseteq \mathbb{N}$, A infinite, $(x_n)_{n\in A}$ is a **subsequence** corresponding to $A = \{n_k | k \in \mathbb{N}\}$

$x_n \to x$ iff every subsequence converges to x. Idea: if $x_n \to x$, points in the subsequence have to be in there at some point (infinite subset of $\mathbb{N}$). If $x \not\to y$ $\exists\, \epsilon > 0$ s.t $\{n \in \mathbb{N} | x_n \notin B_\epsilon(y)\}$ is infinite.

Theorem If x limit point of range (x_n) then x is a subsequential limit of (x_n) (Converse false)
Idea: $x_{n_k} \subset B_{1/k}(k)$. Make some $n_k > n_{k-1}$. This also implies that if (x_n) sequence in compact space K, (x_n) has a convergent subsequence.

Bolzano Weirstruass Bounded sequences in $\mathbb{R}^n$ have convergent subsequences.

Theorem For any seq(x_n), set of all subsequential limits is closed. (but not because E' is closed). Not all subsequential limits are limit points.

Proof: Let $\{A_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ be infinite indexing sets (used to create subsequences) and y_i be a corresponding subsequential limit ($x_n \xrightarrow{A_i} y_i$) with $y_i \to y$. Let A_i^j be the $j-1$th element of the $i-1$th subsequence indexing set and $h_0 = A_0^0$. Define a subsequence by, given n_{k-1}, taking $n_k \in A_k$ $(n_k > n_{k-1})$ s.t $d(x_{n_k}, y_k) < 2^k$.

$$d(x_{n_k}, y) \le d(x_{n_k}, y_k) + d(y_k, y) \le 2^{-k} + \epsilon/2 < \epsilon$$

if you fix $\epsilon > 0$ and take $N \in \mathbb{N}$ s.t $2^{-N} < .5\epsilon$, with $d(y_k, y) < .5\epsilon$ by the definition of a limit.

Sequences in $\mathbb{R}^d$. $\overrightarrow{x}_n \to \overrightarrow{x}$ iff $\lim x_n^i = x^i$ for $1 \le i \le d$. If $\overrightarrow{x}_n \to \overrightarrow{x}$ (likewise for y), then $(\overrightarrow{x} + \overrightarrow{y}) = \overrightarrow{x} + \overrightarrow{y}$. Same for inner product, scalar multiplication.

Monotone sequences. $a_n \le a_{n+1}$ monotone increasing, and monotone decreasing for $\ge$. Strictly monotone if the inequality is not strict.

Theorem Monotone sequence has a limit iff its bounded

The extended real numbers are not a metric space. If $x_n \to \pm\infty$ say it diverges.

A sequence in $\mathbb{R}$ is *unbounded* iff it has a subsequence tending to $\pm\infty$

For $E_N = \{x_n | n \ge N\}$, $(\sup E_n)_n$ is decreasing and $(\inf E_n)_n$ is increasing.
Define $\limsup x_n = \overline{\lim}\ x_n = \lim(\sup E_n)$ and $\liminf x_n = \underline{\lim}\ x_n = \lim(\inf E_n)$

$\limsup x_n$, if finite, is least number α s.t $\forall \epsilon > 0\ \exists N$ s.t $n > N \implies x_n < \alpha + \epsilon$

Theorem Let y be the subsequential limit of x_n. Then $y \in [\liminf x_n, \limsup x_n]$

Proof: $y - \limsup x_n = \epsilon > 0$. Then $\exists N$ s.t $\sup\{x_n | n > N\} < \limsup x_n + .5\epsilon$. Thus $n > N \implies x_n < y - .5\epsilon$ and $B_{.5\epsilon}(y)$ contains at most finitely many $(\le N)$ x_n (contradiction). Analogous for $\liminf$.

$\liminf x_n \le \limsup x_n$. (If bounded, B-W y exists. For unbounded, do cases)

For (x_n) in $\mathbb{R}$, $\exists A, B \subseteq \mathbb{N}$ s.t $x_n \xrightarrow{A} \limsup x_n$ and $x_n \xrightarrow{B} \liminf x_n$

Proof: Fix $\epsilon > 0$. Let $\alpha = \limsup x_n$ finite. Take $n_0 = 0$ and given n_{k-1} take $N > n_k$ s.t $\sup\limits_{n \ge N} x_n \in B_\epsilon(\alpha)$. Then take $n_k > N$ s.t $x_{n_k} \ge \sup\limits_{n\ge N} x_n - \epsilon$. If $a = \infty$, range(x_n) not bounded above, has subsequence that limits to ∞, similar for bounded above.

If (x_n) in $\mathbb{R}$, E is a set of subsequential limits in $\mathbb{R}^k$, then $\limsup x_n = \sup E$ and $\liminf x_n = \inf E$.

Proof: let $\alpha = \limsup x_n$. By previous $\alpha \in E$, so $\alpha \leq \sup E$. WTS $\alpha \geq \sup E$. $\forall N \in \mathbb{N}$, let $y_N = \sup_{n \geq \mathbb{N}} x_n \in \mathbb{R}^K$. $\forall n \in \mathbb{N}$, $y_n \geq x_n$. Also $y_n \to \alpha$ by definition, all subsequential limits are cinvergent to α as well. If $x_n \vec{A} x \in \mathbb{R}^K$ then $x \leq \alpha$

3.1 Series

Given (x_n) in $\mathbb{R}^d$ and $N \in \mathbb{N}$, denote $S_N = \sum_{n=0}^N x_n$ partial sums. Define infinite sum by $\lim S_N$.

Theorem If the infinite sum conv (*converges* - limit of partial sums exists), then $x_n \to 0$

Converse of above theorem false – the harmonic series $\sum 1/n$ diverges. One way to see this is that the partial sums aren't Cauchy: $|S_{2N} - S_N| = |\sum^{@N} 1/n - \sum^N 1/n| = \sum_{N+1}^{2N} 1/n \geq .5N \sum_{N+1}^{2N} 1 = .5$

If $\sum a_n = A$ and $\sum b_n = B, \sum a_n + b_n = A + B$.

Theorem For $r \in \mathbb{C}, |r| < 1$ yields $\sum_{n=0}^\infty r^n = \frac{1}{1-r}$ (diverges otherwise)

Theorem If $p > 1$, $\sum 1/n^p$ converges; diverges otherwise

Comparison test: let $(a_n), (b_n) \in \mathbb{R}^d$ with $0 \leq ||a_n|| \leq b_n$. $\sum b_n$ convergent $\implies \sum a_n$ convergent.

Proof: set $A_N = \sum^N a_n$. Fix $\epsilon > 0$. Since $\sum b_n$ cauchy, $\exists N$ s.t $\sum_{n=p}^q b_n < \epsilon \ \forall N \leq p \leq q$. So $\forall M > N$

$$||A_M - A_N|| = ||a_M + \dots a_{N+1}|| \leq \sum_{i=N-1}^M ||a_i|| \leq \sum_{n=p}^q b_n < \epsilon$$

Theorem If $(a_n), (b_n) \in \mathbb{R}$ s.t $0 \leq a_n \leq b_n$ and $\sum a_n$ divergent, then $\sum b_n$ divergent.

$n^2 \leq 2^n$. Every geometric series is bounded by $\sum 1/N$ for N sufficiently large.

(Limit comparison test) Let $(a_n), (b_n) \in \mathbb{R}^+$. If the limsup of $a_n/b_n < \infty$ and b_n converges, then $\sum a_n$ converges. If the lim inf of the ratio is positive and $\sum b_n$ diverges, so does $\sum a_n$.

Proof: $\limsup \frac{a_n}{b_n} = R$. Then $\exists N$ s.t $n \geq N \implies \frac{a_n}{b_n} \leq R + 1$. So $a_n < (R+1) b_n$; use comparison test.

Intuition from limit comp test only goes so far: given $\sum x_n$ divergent $\exists (a_n)$ s.t $\limsup \frac{a_n}{x_n} = 0$ and $\sum a_n$ diverges. Given $\sum y_n$ convergent, $\exists (b_n)$ s.t $\liminf \frac{b_n}{y_n} = \infty$ and $\sum b_n$ converges. See HW6 #4 (partial sums).

Big/little o notation: $a_n = O(b_n)$ if $\limsup a_n/b_n < \infty$; $a_n = o(b_n)$ if $\limsup a_n/b_n = 0$

(Ratio Test) If $\limsup a_{n+1}/a_n < 1$ then $\sum a_n$ convergent. If > 1, divergent

Proof: Let $\limsup a_{n+1}/a_n = \lambda < 1$. Take $r = .5(1 + \lambda)$. Then $\exists N$ s.t $\forall n > N$ $a_{n+1}/a_n < r$, so $a_{N+M} \leq r^M a_N$ and $\sum_{n=N}^\infty a_n \leq \sum_{n+N}^\infty a_n r^n$ convergent.

(Root Test) Given $\sum a_n$, set $\lambda = \limsup (a_n)^{1/n}$. Then $\sum a_n$ convergent if $\lambda < 1$, divergent if > 1.

Proof: If $\lambda < 1$, set $r = .5(1 + \lambda)$. $\exists N$ s.t $\forall n \geq N$ $a_n \leq r^n$, convergent

Note =1 ambiguous for both, but the root test strictly better. Suppose $\lambda < \infty$ is the ratio test value. Then $\exists N$ s.t $a_{n+m} \leq (\lambda + \varepsilon)^m a_N$. So the root test value is less than $\lambda + \varepsilon$ and thus less than ratio test value.

$\sum a_n$ *converges absolutely* if $\sum ||a_n||$ conv. Convergent $\sum a_n$ *converges conditionally* if $\sum ||a_n||$ doesn't conv

Abs conv stronger than conv and our tests don't like conditional conv (cc always yields root test = 1)

(Alt Series Test) Let (a_n) be a decreasing, non-negative sequence with $\lim a_n = 0$. $\sum (-1)^{n+1} a_n$ conv

Proof: $S_{2N} = (a_1 - a_2) + (a_3 - a_4) + \dots$, $S_{2N+1} = a_1 - (a_2 - a_3) - (a_4 - a_5)$. $0 \leq S_{2N} \leq S_{2N+1} \leq a_1$. Bounded so both conv by monotone convergence. Limit of their difference is the limit of a_{2N+1}, which is 0.

Say $\sum x_n$ conditionally convergent. The sum of only its positive terms is unbounded (same with negative). Idea: divide the terms between $a_n > 0$ and $b_n < 0$. Thus $\sum |x_n| = \sum a_n - \sum b_n$ diverges. $\sum a_n$ and $\sum b_n$ can't both converge, in fact both diverge.

Define $k(n) : \mathbb{N} \to \mathbb{N}$ 1-1. Then for $b_n = a_{k(n)}$, $\sum b_n$ is a *rearrangement* of $\sum a_n$

Theorem If $\sum a_n = A$ converges absolutely, then every rearrangement of $\sum a_n$ also converges to A.

Proof: Let r be the rearrangement bijection and fix $\epsilon > 0$. Then $\exists N$ s.t $\sum_{n=N}^{\infty} |a_n| \leq \epsilon$. Take $M = \max_{i \in [1,N]} r(N)$. Then $|\sum_{i=M}^{\infty} a_{r(n)}| \leq \sum_{n=N}^{\infty} |a_n| < \epsilon$.

Theorem (Riemann rearrangement) If $\sum x_n$ is conditionally convergent, there exists rearrangements where the rearranged series converges to $\pm\infty$ and any real number.

Idea: Let A and B be the set of indices of pos/negative terms respectively. Define $a_n = x_m$, where m in the nth term in A, and $b_n = -x_m$ similarly. (e.g. $A = \{3, 7, 11..\} \to (x_3, x_7, x_{11})$). Know $x_n \to 0$, so same for a_n, b_n. Take N s.t $n > N \implies b_n < .5$. Take t_k s.t $\sum_{n=1}^{t_k} a_n > k$. (bc sum diverges). For $n > t_k + N + k$, partial sum is greater than $C - k + k/2 \to \infty$ (sum of b_1 to b_N is C). For $\lambda \in \mathbb{R}^+$, take # of a_n to get beyond λ, then minimum number of b_n to go back, and so on. Similar consideration for $\lambda \in R^-$

Let $(a_n) \in \mathbb{C}$. A **power series** is a function which assigns to $z \in \mathbb{C}$ the series $\sum a_n z^z = a_0 + a_1 z + \dots$. For $D = \{z | \sum a_n z^n \text{ convergent}\}$, the power series defines a function $D \to \mathbb{C}$.
Notable Examples: $\sum z^n/n! = e^z$, $(-1)^n z^{2n+1}/(2n+1)! = \sin(z)$

Theorem For $\sum a_n z^n$ power series, $\exists R \in [0, \infty]$ s.t $\sum a_n z^n$ converges absolutely in $|z| < \mathbb{R}$ and diverges otherwise. Define *radius of convergence* as $R = \limsup |a_n / a_{n+1}|$ (if converges, otherwise $R = (\limsup |a_n|^{1/n})$

Convergence on boundary nebulous; if the components converge then the whole thing does

For $p \in [1, \infty]$, the **p-norm** on $\mathbb{R}^n$ is $||\vec{x}||_p = (|\sum x_i|^p)^{\frac{1}{p}}$.

Theorem For any $x \in \mathbb{R}^d$ and $1 \leq p \leq q \leq \infty$ $||x||_q \leq ||x||_p \leq d^{\frac{1}{p} - \frac{1}{q}} ||x||_q$.

Define the metric space (X, d) to be with respect to the p-norm. Most properties (open, convergent, compact, cauchy, connected) hold in the 2-norm iff they hold in the p-norm

Infinite dimensional vector spaces: Let $\cup_{n=1}^{\infty} \mathbb{R}^n = \mathbb{R}^{\mathbb{N}}$
Idea: $\mathbb{R} \subseteq \mathbb{R}^2$. So generalizing this further, $n < m \implies \mathbb{R}^n \subseteq \mathbb{R}^m$ by $(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$.

Let $p \in [0, \infty]$ with $q \in \mathbb{R}$. In $(\mathbb{R}^{\mathbb{N}}, p)$, $(\frac{1}{n^q})$ is Cauchy if $pq > 1$ and unbounded if $pq \leq 1$. Bounded, not Cauchy if $pq = \infty \cdot 0$.

Proof: $||a_n||_p = \left[\sum_{i=1}^{n} \frac{1}{i^n}\right]^{1/p}$ and for $n < m$ $||a_m - a_n||_p = \left[\sum_{i=n+1}^{m} \frac{1}{i^n}\right]^{1/p}$. For $p = \infty$, $||a_n||_\infty$ is 1 if $q \geq 0$ and n^{-q} otherwise. So bounded iff $q \geq 0$. $||a_n - a_m||_\infty = \max\{n^{-q}, m^{-q}\}$, so Cauchy iff $1/p \to 0$

Theorem For $p \in [1, \infty]$ $(\mathbb{R}^{\mathbb{N}}, p)$ is not complete.

$\mathbb{R}^\infty$ is the set of all sequences in $\mathbb{R}$. Identify $(x^1, \dots, x^d) \in \mathbb{R}^{\mathbb{N}}$. Now we can formally consider $\mathbb{R}^{\mathbb{N}}$ sequences with finitely many non-zero terms. Moreover, $\mathbb{R}^\infty$ is a vector space

For $p \in [1, \infty], a \in \mathbb{R}^\infty$, denote $||a||_p = |\sum_{n=0}^{\infty} (a^n)^p|^{1/p} \in \mathbb{R}^*$ and $||a||_\infty = \sup_n |a^n| \in \mathbb{R}^*$.
For $p \in [1, \infty]$, denote $\ell^p - \{a \in \mathbb{R}^\infty \big| \, ||a||_p \leq \infty\}$. Then ℓ^p is a metric space with p-norm.
And for any $p \in [1, \infty], \mathbb{R}^{\mathbb{N}} \subseteq \ell^p \subseteq \mathbb{R}^\infty$

Let $a \in \mathbb{R}^\infty$, $a_n = (a^0, \dots, a^n, 0, 0, \dots) \in \mathbb{R}^\mathbb{N}$. Then $a \in \ell^p$ $(p < \infty)$ iff $a_n \to a$ in ℓ^p

Sketch: $a \in \ell^p$ iff $\sum |a^n|^p$ conv. $a_n \to a$ iff $\sum_N^\infty |a^n|^p \overset{N}{\to} 0$

Theorem For $p \in [1, \infty)$, ℓ^p is the completion of $\mathbb{R}^\mathbb{N}$ with the ℓ^p norm.

Proof: Let $(a_n) \in \ell^1$ be Cauchy. WTS $\exists a \in \ell^1$ s.t $a_n \to a$. Let a_n^k be the kth element of the nth sequence. Let $a_n^k \overset{n}{\to} a^k \in \mathbb{R}$ (can do this since because partial sums will be cauchy). Now define $a = (a^1, a^2, \dots) \in \mathbb{R}^\infty$. We know it's in ℓ because we can define R strictly greater than its norm (so will be greater than sum of first k components for all k). Cannot take straightforward limit because each k needs different N. However, we can define a N with respect to $||a_n - a_m||$, use an absolute sum taking a limit in m, then taking an infinite sum with sup over all n, giving us what we want.

4 Continuity

Let $f : X \to Y$ a function. we denote Y the codomain for $E \subseteq X$

$f(x)$ is the **image**. For $E \subset Y$, the **inverse** image $f^{-1}(E) = \{x \in X | f(x) \in E\}$

For $f : E \to y$ with $p \in E'$ and $E \subset X$, say $\lim\limits_{x \to p} f(x) = y$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t $x \in E$ with $x \neq p$,

$$d_X(x, p) < \delta \implies d_Y(f(x), y) < \varepsilon$$

Let $S^1 = \{z \in \mathbb{C} | |z| = 1\}$ (unit circle) and define $\arg : S^1 \to [0, 2\pi)$. $\lim_{z \to 1} \arg$ DNE.
$\forall \delta > 0$, $e^{i\frac{\delta}{2}}, e^{i(2\pi - \frac{\delta}{2})} \in B_\delta(1)$. But $d(x, y) = 2\pi > \varepsilon$

$p \in E'$, $\lim\limits_{x \to p} f(x) = y$ iff $\forall (x_n) \in E$, $\lim_n x_n = p$, $x_n \neq p$ $\forall n \implies \lim f(x_n) = y$

Proof: $(\impliedby)$ Suppose the limit is not y: then $\exists \varepsilon > 0$ s.t $\forall \delta$ $\exists x \in B_\delta(p)$ where $d(f(x), y) \geq \varepsilon$. For each n, take $x_n \in B_\delta(p)$ s.t $d(f(x_n), y) \geq \varepsilon$. Then $f(x_n)$ does not converge to y

So limits are unique: a and b both $\lim\limits_{x \to p} f(x) = y$, $a = b$. "Vacuously" continuous at isolated points

f is **continuous** at $p \in E$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t $\forall x \in E$ where $d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \varepsilon$

Earlier limit definition: undefined if $p \notin E'$. Continuity guarantees everything in a small enough nbhd is a lp

Theorem $f : X \to Y$ continuous iff $\forall U \subset X$ open $\implies f^{-1}(U) \subset X$ open

f continuous iff pre-images of closed sets are closed

Theorem Composition of continuous functions are continuous
WTS $X \overset{f}{\to} Y \overset{g}{\to} Z$

Proof: (1): Fix $\varepsilon > 0, p \in X$. Since g continuous, $\exists \eta > 0$ s.t $d_Y(f(p), y)) < \eta \implies d_Z(g \circ f(p), g(y)) < \varepsilon$. Since f continuous, $\exists \delta > 0$ s.t $d_X(p, x) < \delta \implies d_Y(f(p), f(x)) < \eta \implies d_Z(\circ, \circ) < \varepsilon$

Proof: (2) Let $x_n \to p \in X$. Then $f(x_n) \to f(p)$. $g(f(x_n)) \to g(f(p))$.

Proof: (3) Let $U \subset Z$ open. Then $g^{-1}(U) \subset Y$ open, so $f^{-1}(g^{-1}(U)) \subset X$ open

If $\lim\limits_{x \to p} f(x) = y$ for $f : E \to Y$, $p \in E'$, and $g : Y \to Z$ cont $\lim\limits_{x \to p} g \circ f(x) = g(y)$

Theorem $+ : \mathbb{R}^2 \to R$ s.t $+(a, b) = a + b$ is continuous

Proof: $+((x_n, y_n)_n) \to x_n + y_n \to x + y = +((x, y))$

A function is bounded if its image is bounded. Continuous function on a compact domain is bounded.

Theorem For $f : K \to Y$ continuous, K compact, $f(K)$ is compact

Proof: Let $\{G_a\}_a$ open cover of $f(K)$. WTS $\{f^{-1}(G_a)\}_a$ open cover of K. Each $f^{-1}(G_a)$ is open. For $x \in K$, $f(X) \in f(K)$ so $\exists \alpha$ s.t $f(x) \in G_\alpha$, so $x \in f^{-1}(G_\alpha)$.
So $\exists A$ finite s.t $K \subset \cup_{a \in A} f^{-1}(G_a)$. $\forall y \in f(K)$ $\exists x \in K$ s.t $y = f(x)$. So $\exists alpha \in A$ s.t $x \in f^{-1}(G_\alpha)$, so $y = f(x) \in G_\alpha$. So $\{G_\alpha\}_{\alpha \in A}$ is a finite subcover.

Let $f : K \to Y$ surjective, K compact. If $U \subset K$ open, then $f(U)$ open. If f bijection, $f^{-1} : Y \to K$ continuous

A function $f : X \to Y$ is **uniformly continuous** if $\forall \varepsilon > 0 \exists \delta > 0$ s.t $d_x(x, y) \implies d_y(f(x), f(y))$

Theorem Let $f : X \to Y$ uniformly continuous, (x_n) cauchy in X. $(f(x_n))$ Cauchy in Y

Proof: Fix $\varepsilon > 0$. $\exists \delta$ s.t $d_x(a, b) < \delta \implies d_Y(f(a), f(b)) < \varepsilon$. Moreover, $\exists N$ s.t $m, n > N \implies d_X(f(x_n), f(x_m)) < \delta \implies d_Y(f(x_n), f(x_m))$

Theorem $f : K \to Y$ continuous, K compact, then f uniformly continuous.

Proof: Fix $\varepsilon > 0$. $\forall x \in K$, define r_x s.t $B_{r_x}(x) \subset f^{-1}(B_{\frac{\varepsilon}{2}} f(x))$, so $\{B_{\frac{r_x}{2}}(x)\}$ open cover. Over finite subcover, take $\delta = \min \frac{r_{x_n}}{2}$. If $d_X(x, y) < \delta$ $\exists x_n$ s.t $d(x, x_n) < \frac{r_{x_n}}{2}$. So $d(x_n, y) < r_{x_n}$ and $d_Y(f(x), f(y)) < \varepsilon$

Theorem Let $E \subseteq X$, $f : E \to Y$, $p \in E$. If f unif cont and Y complete, then $\lim_{x \to p} f(x)$ exists.

Proof: Let $(a_n) \in E$, $a_n \to p$, $a_n \neq p$ $\forall n$. Then (a_n) cauchy, $(f(a_n))$ Cauchy, limit exists call it y. Let $(b_n) \in E$, $b_n \to p$, $b_n \neq p$ $\forall n$. Then $(a_0, b_0, a_1, b_1, \dots) \to p$, cauchy. By thrm the seq-function converges to y

Theorem Let $E \subseteq X$. $f : E \to Y$ uniformly continuous, Y complete. Then $\exists \overline{f} : \overline{E} \to Y$ continuous where $f(x) = \overline{f}(x)$ $\forall x \in E$ (see HW for proof)

Theorem $f : X \to Y$ continuous, X connected, then $f(X)$ connected

Proof: Proof by contradiction: $A, B \subseteq Y$, $f(x) \subseteq A \cup B$, $\overline{A} \cup B = \overline{B} \cup A = \emptyset$. Then $X \subseteq f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A), f^{-1}(B)$ disjoint.

(IVT) $f : X \to \mathbb{R}$ continuous,X connected , $\exists a, b \in X$ s.t $r \in (f(a), f(b)) \implies \exists c$ s.t $f(c) = r$

Theorem E connected iff $\forall f : E \to \{0, 1\}$ continuous $\implies f(E) = 0$ or $f(E) = 1$

Proof: ($\Longleftarrow$) E disconnected, so $E = A \cup B$, disjoint, clopen, nonempty.

A set E is *path-connected* iff $\forall a, b \in E$ $\exists f : [0, 1]] \to b$ continuous s.t $f(0) = a, f(1) = b$.
Theorem Path-connected iff connected

For $f : \mathbb{R} \to \mathbb{R}$, $p \in \mathbb{R}$, define $g : (-\infty, p) \to \mathbb{R}$ and $h(p, \infty)$ equal to f on their domains. limit "from the right" $x \to p^+$ is defined as $\lim_{x \to p} h(x)$, similarly "from the left" $x \to p^-$ with g. Cont iff limits equal

"Simple" *discontinuity*: removable ("hole") & jump (e.g piece wise). "Essential": limit from left or right DNE

Theorem Monotone functions can't have infinitely many or non-jump discontinuities

5 Derivatives ($f : \mathbb{R} \to \mathbb{R}$)

Derivative of f at $a \in \mathbb{R}$ is $f'(a) = \frac{d}{dx} f(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

If f derivative exists $\forall a \in U$ open, f is *differentiable* on U

Theorem If f is differentiable at a, then f is continuous at a
 Proof: Since $f'(a)$ exists, $\lim_{x \to a} f(x) - f(a)$

Theorem (Prod/Quotient Rules) $(fg)'(p) = f'(p) \cdot g(p) + f(g) \cdot g'(p)$ and $(1/f)'(p) = -f'(p)/f(p)^2$

Theorem (Chain Rule) $f : U \to \mathbb{R}, f(U) \subset V$ open,$g : V \to \mathbb{R}$. Assuming existance, $(g \circ f)'(p) = g' \circ f(p) \cdot f'(p)$

Theorem (Rolle's) $f : [a, b] \to \mathbb{R}$ cont and diff. If $f(a) = f(b)$, then $\exists c \in (a, b)$ s.t $f'(c) = 0$

(MVT) Corollary to Rolle's without $f(a) = f(b)$ condition: $\exists c \in (a, b)$ s.t $f'(c)(b - a) = f(b) - f(a)$

$f : X \to Y$ is **Lipschitz**:if $\exists M > 0$ s.t $\forall x, y \in X d_Y(f(x), f(y)) \leq M d_X(x, y)$

Theorem For $f : U \to \mathbb{R}$ diff w/ U cnctd, if $\exists M \geq 0$ s.t $|f'(x)| \leq M \forall x \in U$, f is lipschitz

Lipschitz gives some information on the rate of convergence, more than just differentiable at a point
Theorem If f diff at p, $\exists M \geq 0, r > 0$ s.t $|f(x) - f(p)| \leq M|x - p| \;\forall x \in B_r(p)$
BUT Lipschitz weaker than continuity in terms of $\implies$ differentiability.;$f(p) + f'(p)(x - p)$ is tangent line.

$a + b(x - p)$ is the *best linear approx* of f at p if $\lim_{x \to p} \left| \frac{f(x)-[a+b(x-p)]]}{x-p} \right| = 0$

Theorem For $f : U \to \mathbb{R}$ cont at $p \in U$ best linear approx iff f diff at p, $a = f(p), b = f'(p)$

With $f^{(n)}(x) = \frac{d^n}{dx^n} f(x)$, for f n-diff at P, nth degree **Taylor polynomial** is $T_p^n(f)(x) = \sum_{i=0}^n \frac{1}{i!} f^{(i)}(p)(x-p)$
As n gets larger, the error between the approximation and the function goes to 0

Let $f(b) = f(a) = f'(a) = ... = f^{n-1}(a) = 0$, $f^n(c) = 0$ (consider Rolle's), then $\exists c$ s.t $f^{(n)}(c) = 0$
Theorem (Taylor) Let $\alpha, \beta \in (a, b)$. $\exists z \in (\alpha, \beta)$ s.t $f(\beta) = T_\alpha^{n-1} + \frac{f^{(n)}(z)}{n!}(\beta - \alpha)^n$

Theorem (L'Hopital) For $f(x) = g(x) = 0$, existence of deriv, and $g'(x) = 0$. $\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$
Theorem If $f : U \to \mathbb{R}$ n-times diff at $p \in U$, and $p : \mathbb{R} \to \mathbb{R}$ degree n-polynomial then $\lim_{x \to o} |f(x) - q(x)|/|(x - p)^n| = 0$ iff $Q = T_p^n(f)$
 Proof: By L'hopital (see Rudin for a stronger statement)

$$\lim_{x \to p} \frac{f(x) - T_p^n(f)(x)}{(x - p)^n} = \cdots = \lim \frac{f^{(n-1)}(x) - f^{(n-1)}(p) - f^{(n)}(p)(x - p)}{n!(x - p)} = 0$$

If Q is best polynomial approximation, add and subtract $T_p^n(f)(x)$ from given limit so $\frac{T_p^n - Q}{(x-p)^n} \to 0$. But $[T_p^n(f) - Q](x) = a_0 + a_1(x - p) + \cdots + a_n(x - p)^n$ then limit converges iff all the coefficients are equal.

The **Taylor series** for f at p is the power series $T_p(f)(x) = \sum \frac{f^{(n)} * p)}{n!}(x - p)^n$.

If $\exists R > 0$ s.t $T_p(f)(x) = f(x) \;\forall x \in B_R(p)$ call f *analytic* at p. Are infinitely diff.

Theorem Let $f : [a, b] \to \mathbb{R}^n$ cont diff on (a, b) w/ $||f'|| \leq M$. $||f(b) - f(a)|| = M|b - a|$
 Proof: Let $v \in \mathbb{R}^n$. By MVT $|v(b - a)^{-1}(f(b) - f(a)| \leq \sup(v \cdot f') \leq ||v||M$. Take $v = f(b) - f(a)$

6 Integration

A **partition** of $[a,b] \subseteq \mathbb{R}$ is a finite set of points $\mathcal{P} = \{x_i\}_{i=0}^n$ s.t $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$

Given partitions P, Q if $P \subseteq Q$, Q is a *refinement* of P. $P \cup Q$ is *common refinement*

Given partition P, if $t_i \in [x_{i-1}, x_i]$ $(i \in [1,n])$, P is a *tagged partition* with tags $\{t_i\}$

Given tagged partition P, let $\Delta_i = x_i - x_{i-1}$, $I_i = [x_{i-1}, x_i]$, and $S(f,P) = \sum_{i=1}^n f(t_i)\Delta_i$ be the Riemann sum. Define $m_i = \inf\{f(x)|x \in I_i\}$ and $M_i = \sup\{f(x)|x \in I_i\}$. Define the upper and lower **Darboux sums** by $L(f,P) = \sum_{i=1}^n m_i\Delta_i$ and $U(f,P) = \sum_{i=1}^n M_i\Delta_i$. Note $L(f,P) \leq S(f,P) \leq U(f,P)$.

Denote the *lower Darboux integral* by $\underline{\int_a^b} f = \sup_P L(f,P)$ and similarly for upper integral. If they are both equal to some $C \in R$, f is **integrable** and $\int_a^b f - C$

Given $\alpha : [a,b] \to \mathbb{R}$ increasing and P part, denote $\alpha_i = \alpha(x_i)$ $(x_i \in P)$. Define $L(f,P,\alpha) = \sum_{i=1}^n m_i[\alpha_i - \alpha_{i-1}]$ and $\underline{\int_a^b} f d\alpha = \sup_P L(f,P,\alpha)$ (similar for $U(f,P,\alpha)$. If equality exists, common value $\int f d\alpha$ is **Stieltjes** integral and say $f \in \mathcal{R}(\alpha)$. Notice $\alpha(x) = x$ yields Darboux. Define $f \in \mathcal{R}$ Darboux integrable.

Let $f : [a,b] \to \mathbb{R}$ bounded, P, Q part s.t. $P \subseteq Q$. Then $L(f,P,\alpha) \leq U(f,Q,\alpha)$ and $L(f,Q,\alpha) \leq U(f,P,\alpha)$

The above result yields for *any* partitions P, Q $L(f,P,\alpha) \leq U(f,Q,\alpha)$
Proof: $L(f,P,\alpha) \leq L(f,P \cup Q,\alpha) \leq U(f,P \cup Q,\alpha) \leq U(f,Q,\alpha)$

$f \in \mathbb{R}(\alpha)$ iff $\forall \varepsilon > 0$ $\exists P$ s.t $U(f,P,\alpha) - L(f,P,\alpha) < \varepsilon$

Theorem If $f : [a,b] \to \mathbb{R}$ cont, α inc, then $f \in \mathbb{R}(\alpha)$
Proof: Let $L = (\alpha(b) - \alpha(a))^{-1}$ and fix $\varepsilon > 0$. $\exists \delta > 0$ s.t $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Let P s.t $\Delta_i < \delta$ $\forall i$. Then $M_i - m_i \leq L\varepsilon$, so $[U - L](f,P,\alpha) \leq L\varepsilon(\alpha_i - \alpha_{i-1}) \leq \varepsilon$

Theorem Let $g : [a,b] \to \mathbb{R}$ bounded, $f : \mathbb{R} \to \mathbb{R}$. If $g \in \mathbb{R}(\alpha)$, f cont, then $f \circ g \in \mathcal{R}(\alpha)$
Proof: If P s.t $[U - L](g,P,\alpha)$ small, then $\forall i$ either $M_i - m_i$ small, or $\alpha_i - \alpha_{i-1}$ small. If $M_i - m_i$ small, $f(M_i) - f(m_i)$ small; f doesn't matter much. WLOG f uniformly continuous (since g bounded). Take ε, δ s.t $\sup_{I_i} g(x) - g(y) < \delta \implies \sup f \circ g(x) - f \circ g(y) < \varepsilon$. $[U - L](g,P,\alpha) \geq \sum_{i \text{ s.t } M_i - m_i > \delta} \delta(\alpha_i - \alpha_{i-1})$. Let $K = \sup f \circ g - \inf f \circ g$. Then $[U - L](f \circ g, P, \alpha) \leq \varepsilon[\alpha(x) - \alpha(y)] + \sum_{M_i - m_i > \delta} K[\alpha_i - \alpha_{i-1}] \leq \varepsilon[\alpha(x) - \alpha(y)] + [U - L](g,P,\alpha)K\delta^{-1}$. This bound holds for all partitions so $\exists P$ s.t $[U - L](g,P,\alpha)K\delta^{-1} = 1$

Theorem Stieltjes integration preserves monotonicity (α incr)

Theorem Let $\phi : \mathbb{R} \to \mathbb{R}$ cont and strictly incr. $f, \alpha : [\phi(a), \phi(b)] \to \mathbb{R}$, $f \in \mathcal{R}(\alpha)$. Then $g = f \cdot \phi$ and $\beta = \alpha \cdot \phi$ yield $g \in \mathcal{R}(\beta)$. Moreover, $\int_a^b g d\beta = \int_{\phi(a)}^{\phi(b)} f d\alpha$
Proof: For P partition of $[a,b]$, $\Phi(P) = \{\phi(x_i)\}$ is a partition of $[\phi(a), \phi(b)]$. $U(f, \Phi(P), \alpha) = U(g, P, \beta)$..

Consider α diff on $[a,b]$. $\alpha_i - \alpha_{i-1} = \alpha_i - \alpha_{i-1} \cdot (\Delta_i/(x_i - x_{i-1})) = \alpha'(t_i)\Delta_i$ (some $t_i \in I_i$)

Theorem Let $f, \alpha : [a,b] \to \mathbb{R}$ with $\alpha' : (a,b) \to \mathbb{R}$ cont. $f \in \mathcal{R}(\alpha)$ iff $f\alpha' \in \mathcal{R}$ and $\int f d\alpha = \int f\alpha'$

If $f(x) = g(x)$ $\forall x \neq p$ then $\int f d\alpha = \int g d\alpha$ if α cont at p

Given $f : [a,b] \to \mathbb{R}$, $\mathcal{P}$ partition, the *variation* of f over $\mathcal{P}$ is $V(f,\mathcal{P}) - \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$. The *total variation* of f is $TV(f) = \sup_{\mathcal{P}} V(f,\mathcal{P})$. If $TV(f)$ is finite, f has **bounded variation**; "f is BV"

If f monotone, $TV(f) = |f(b) - f(a)|$. So the sum of montone functions (on a closed interval) is BV

Theorem (Jordan Decomposition) f is BV iff its equal to the sum of inc and dec fun.

Proof: ($\implies$) $f_t = f$ restricted to $[a.t]$. Then $TV(f_x)$ increasing. Given $c < d$, let $\mathcal{P}$ partition of $[c, d]$. $f(d) - f(c) = \sum f(x_i) - f(x_{i-1}) \leq \sum |f(x_i) - f(x_{i-1})| = TV(f_d) - TV(f_c)$. $f(d) - TV(f_d) \leq f(c) - TV(f_c)$. So $f(x) = TV(f_x) + [f(x) - TV(f_x)]$, a sum of incr and decr.

Let $f, \alpha : [a, b] \to \mathbb{R}$, α BV, $\mathcal{P}$ partition s.t. $M_i - m_i < \varepsilon$ with $\{t_i\}, \{s_i\}$ tags. Then $|\sum f(t_i)[\alpha_i - \alpha_{i-1}] - \sum f(s_i)[\alpha_i - \alpha_{i-1}]| \leq \varepsilon TV(\alpha)$. So integrate with BV weights if α monotonic. If α (BV) not monotonic, use JD such as: $\alpha = \alpha_+ - \alpha_-$, incr and decr function. Define $\int d\alpha = \int f d\alpha_+ - \int f d\alpha_-$. So $\mathcal{R}(\alpha) = \mathcal{R}(\alpha_+) \cap \mathcal{R}(\alpha_-)$. Formalizing this, let α BV. If $f \in \mathcal{R}(\alpha)$, $\forall \varepsilon > 0$ $\exists \mathcal{P}$ tagged partition s.t $|\int f d\alpha - \sum (t_i) p[\alpha_i - \alpha_{i-1}]| < \varepsilon$

Proof: Use use above decomposition, triangle ineq, and then $\mathcal{P}$ that works for both α_+, α_-

(FTC 1) Let $f \in \mathcal{R}$ on $[a, b]$ and $F(x) = \int_a^x f$. For, F cont and f cont at p, $F'(p) = f(p)$

Proof: Since f bounded, $|f| \leq M$ (some M, so $|\int_a^x f - \int_a^y f| = |\int_y^x f| \leq M|x - y|$, so Lipschitz continuous. So $F(x) - F(p) = \int_p^x f = \int_p^x f(p) + \int_p^x [f - f(p)] = f(p)(x - p) + E$. So $f(p) = f'(p)$ iff $E = o(x - p)$. If f cont at p, for $\varepsilon > 0$ $\exists \delta > 0$ s.t $x = B_\delta(p) \implies |f(x) - f(p)| < \varepsilon$ and $|E| \leq \varepsilon |x - p|$

(FTC 2) Let $f \in \mathcal{R}$ on $[a, b]$ and $f : [a, b] \to \mathbb{R}, f' \in \mathcal{R}$. Then $f(b) = f(a) + \int_a^b f'$

Proof: By MVT, $\exists$ tags of P s.t $\sum f'(t_i)\Delta_i = \sum f(x_i) - f(x_{i-1}) = f(b) - f(a)$. Thus $L(f', f) \leq f(b) - f(a) \leq U(f', f)$. Assume $L - U \to 0$

(Int by Parts) For $f.g : [a, b] \to \mathbb{R}$ w/ $f', g' : [a, b] \to \mathbb{R}\,(\in \mathcal{R})$, $\int f \cdot g' = fg|_b^a - \int f' \cdot g$

Proof: Product rule + FTC: $\int (f \cdot g)' = \int f \cdot g' + \int f' \cdot g = f(b)g(b) - f(a)g(a)$

To bring Stieltjes in (doesn't apply to FTC in general), consider $f \in \mathcal{R}(\alpha)$, f diff. For P partitions

$$\sum [(f\alpha)_i - (f\alpha)_{i-1}] = \sum f_i[\alpha_i - \alpha_{i-1}] + \sum [f_i - f_{i-1}]\alpha_{i-1} =\| + \sum \alpha_{i-1} f'(t_i)\Delta_i \approx \int f d\alpha + \int \alpha f'$$

Theorem $f : [a, b] \to \mathbb{R}$, diff on (a, b), α BV. If $f \in \mathcal{R}(\alpha)$, $f'\alpha \in \mathcal{R}$, then Int by parts holds ($\int_b^a f d\alpha$)
Stieltjes integral is roughly weighted Riemann int of derivative

7 Sequences of Functions

For X, Y metric spaces, $\mathcal{F}(X, Y)$ denote set of all $X \to Y$.

For (f_n) seq in $\mathcal{F}(X, Y)$, say $f_n \to f$ **pointwise** if $\lim f_n(x) = f(x) \ \forall x$

For (f_n) seq in $\mathcal{F}(X, Y)$, say $f_n \to f$ **uniformly** ($f_n \rightrightarrows f$)if $\forall \varepsilon > 0 \ \exists N$ s.t $n \geq N \implies d(f_n(x), f(x)) < \varepsilon$ ($\forall x \in X$)

$f_n \rightrightarrows f$ iff $\sup_{x \in X} d(f_n(x), f(x)) \to 0$

Theorem (Cauchy Criteria): Let Y complete, $(f_n) \in \mathcal{F}(X, Y)$ s.t $\forall \varepsilon > 0, \exists N$ s.t $n, m > N \implies \sup d(f_n, f_m) < \varepsilon$. Then $\exists f \in \mathcal{F}(X, Y)$ s.t $f_n \rightrightarrows f$

Proof: For x fixed, $f_n(x)$ cauchy, lim exists, call it $f(x)$. Fix $\varepsilon > 0$. Choose N s.t $m, n > N \implies \sup_x d(f_n(x), f_m(x)) < \varepsilon$. Then $d(f_n(x), f(x)) = \lim_{m \to \infty} d(f_n(x), f_m(x)) \leq \varepsilon$ ($\forall x$) holds for $n > N_\varepsilon$, independent of x.

Theorem (Weierstrass M-test) For $f_n \in \mathcal{F}(X, \mathbb{R})$, if $\forall n \exists M_n \in \mathbb{R}$ s.t $\sup_x |f_n| \leq M_n$ with $\sum M_n < \infty$. Then $\sum f_n$ converges uniformly.

Theorem Let $E \subseteq X$, $f_n, f : E \to Y$, $p \in E'$. If $f_n \rightrightarrows f$ and $\lim_{x \to p} f_n(x) = L_n \in Y$, then (L_n) cauchy and if $\lim L_n$ exists, $\lim L_n = \lim_{x \to p} f(y)$. If $f_n : X \to Y$ cont and unif conv, then f cont.

Proof:

$$d(L_n, L_m) \leq d(L_n, f_n(x)) + d(L_m, f_m(x)) + \sup d(f_n, f_m)$$

Choose N s.t 3rd term $< \varepsilon$. Choose x s.t $1/2 < \varepsilon$ (dependent on n, m), so cauchy. Define $L = \lim L_n$

$$d(f(x), L) \leq \sup d(f, f_n) + d(f_n(x), L_n) + d(L_n, L)$$

Let X, Y metric spaces w/ Y cplt. Define $C^0(X, Y)$ the set of bounded, cts $X \to Y$ w/ $d_0 = \sup_x d_Y(f(x), g(x))$

C^0 is complete and $\int$ is a continuous function $C^0([a, b]) \to \mathbb{R}$

Let α BV, $f_n \rightrightarrows f$, $f_n \in \mathcal{R}(\alpha)$. $f \in \mathcal{R}(\alpha), \int f_n d\alpha \to \int f d\alpha$

Proof: WLOG α increasing. Let $\varepsilon > 0$, take n s.t $\sup |f_n - f| < \varepsilon$. Take P s.t $U - L < \varepsilon$. $\sup f \leq \sup f_n + \varepsilon$ (over each i, similar for a lower bound with inf).

$$[U - L](f, P, \alpha) = \sum (M_i - m_i)(\alpha - \alpha_{i-1}) \leq (M_i^n - m_i^n + 2\varepsilon)(\alpha - \alpha_{i-1}) \leq \varepsilon + 2(\alpha(b) - \alpha(a))\varepsilon$$

Consider $f_n(x) = f_n(a) + \int_a^x f_n' \to f(a) + \int_a^x f' = f(x)$ ($f_n' \in \mathbb{R}$, $f_n \to f$ pointwise & $f_n' \to g$ uniform, f' cont). So $f'(x) = g(x)$ if g continuous

Let (f_n) diff, $f_n \to f$, $f_n' \rightrightarrows f'$, then $\forall p$, $\frac{f_n(x) - f_n(p)}{x - p} \rightrightarrows \frac{f(x) - f(x)}{x - p}$

Theorem Let (f_n) diff, $f_n \to f$, $f_n' \rightrightarrows g$. $g = f'$

A *modulus of continuity* is $W : [o, \infty) \to [0, \infty]$ s.t $W(0) = \underset{x \to 0}{W(x)} = 0$. f has a modulus of continuity at p if $\exists \omega_p$ (modulus) s.t $d(x, p) < \delta \implies d(f(x), f(p)) < W(\delta)$. f is continuous at o iff f has a modulus of continuity at p. Uniform continuous if modulus exists that is independent of p

A collection $F \subset \mathcal{F}(X, Y)$ is **equicontinuous** if $\forall p \in X \ \exists \omega_p$ s.t $\forall f \in F$, ω_p (modulus) for f at p. Equivalently, if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t $d(x, p) \leq \delta \implies d(f(x), f(p)) < \varepsilon$.

Sequence converges pointwise to a function if for every $x \in D$ and $\varepsilon > 0$ $\exists N > 0$ s.t $n > N \implies |f_n(x) - f(x)| < \varepsilon$

Let $f_n = \cos(x/n)$. Fix $x \in [0, \pi]$. Since $cos(z)$ is continuous, for any $\varepsilon > 0 \exists \delta > 0$ s.t $z \in (0, \delta) \implies |cos(z) - 1| < \varepsilon$. So equivalently $\exists N$ s.t $n > N \implies x/n \in (0, \delta)$. Not a type of continuity, for seq.

x^2 cont with $\omega_p(\delta) = 2|p|\delta + \delta^2$

Theorem If $f_n \to f$ pointwise, (f_n) equicont, then f cont

$\overline{B_2(0)} \subseteq C^0(\mathbb{R})$ is not compact. Take $f_n \in B_2(0)$, $f_n(1/n) = 1$, limit to 0. f_n can't have uniform limit because $f_n \to 0$ not uniform, same with subsets. The problem is C^0 is that convergence is not just about size, but issues with continuity.

Uniform limit of f_n being Lip does not imply that that f_n has bounded lip constant, f_n can have arbitrarily, oscillating steps. But if $f_n \rightrightarrows f$ in $C^0(X, Y)$, then $\forall p$ $\exists \omega_p$ mod that holds at p for f, all f_n

<u>Proof:</u> Fix $p \in X$. Let $\omega_{p,n}$, ω_p be sharp mod at p for f_n, f. Define $\overline{\omega_p}(r) = \sup\{\omega_p(r), \omega_{p,n}(r)\}_{n \in \mathbb{N}}$. By construction, $d(x, p) < \delta \implies d(g(x), g(p)) < \overline{\omega_p}(\delta)$. WTS $\lim \overline{\omega_p}(r) = 0$. Fix $\varepsilon > 0$. Since $f_n \rightrightarrows f$ $\exists N$ s.t $n > N \implies d(f_n(x), f_n(p)) < d(f(x) + f(p) + \varepsilon$. WLOG take $\omega_{p,n}(r) \le \omega_p(p) + \varepsilon$ $(n > N)$. Everything less than n going to 0.

$n^{-1}\sin(n^2 x) \rightrightarrows 0$, $f_n' = ncos(n^2 x)$ not bounded. $|n^{-1}(n^2(x + \delta) - n^{-1}\sin(n^2 x)| \le \min(n\delta, 2n^{-2}) \le 2\delta$

Theorem If $F \subseteq C^0$ compact, then $\exists M \in \mathbb{R}^+$ s.t $|f(x)| \le M$, $f \in F$ equicont

Theorem (Arzela-Ascoli) Let K compact, $F \subseteq C^0(K, \mathbb{R})$. If F bounded in C^0 and F equicont, then any seq in F has conv subseq in C^0

<u>Proof:</u> Let (f_n) seq in F. Let ω_p unif mod for F. $\forall n \in N$ $\exists$ finite collection p_i, δ_i s.t $\omega_{p_i}(\delta_i) < n^{-1}$. $K \subseteq B_{\delta_i}(p_i)$. Take E_n set of all such p_i. Since E_n finite, $\cup E_n$ countable, $\exists A \subseteq \mathbb{N}$ infinite s.t $\lim_A f(p)$ conv $(p \in \cup E_n)$. WTS if f_n conv unif along A. Fix $\varepsilon > 0$. $E = E_n$, s.t $n^{-1} < \varepsilon$, so $\omega_p(\delta_p) \le \varepsilon$ $(p \in E)$. For $x \in K$, $\exists p \in E$ s.t $x \in B_{\delta_i}(p)$. For $m, n \in A$,

$$|f_n(x) - f_m(x)| \le |f_n(p) + f_m(p)| + |f_m(x) - f_m(p)| + |f_n(x) - f_m(p)| \le 2\varepsilon + |f_n(p) - f_m(p)|$$

Theorem For $f : [a, b] \to \mathbb{R}$ cont, $\exists (P_n)$ polynomials s.t. $P_n \rightrightarrows f$

An **algebra** A is a collection of functions s.t. for $f, g \in A$, $f + g \in A$, $fg \in A$, $\lambda f \in A$.

A (function) collection F separates points if $\forall x \ne y$, $\exists f \in F$ s.t $f(x) \ne f(y)$

Theorem (Stone-Weirstrauss) K compact, $A \subseteq C^0(K)$ algebra. If A separates points, it's dense $(C^0(K))$

<u>Proof:</u> First note from HW, $\exists (P_n) \rightrightarrows |x|$. So taking $.5f + g + .5P_n(f - g)$, we have $\min, \max \in \overline{A}$. Take $g \in A$ and define $\overline{g} = (g - g(y))(g(x) - g(y))^{-1} \in A$, $\overline{g}(x) = 1$, $\overline{g}(b) = 0$, $f(x)\overline{g} + f(y)(1 - \overline{g}) \in A$. Fix $x \in K$, $\varepsilon > 0$. $\forall y \in K$, $\exists g_y$ s.t $g_y(x) = f(x)$,$g_y(y) = f(y)$, and cont. So $\exists \delta_y$ s.t $|f - g_y| < \varepsilon$ on $B_\delta(y)$. $K \subseteq \cup_{i=1}^n B_{\delta_i}(y_i)$. $\exists \overline{g}_x \in A$, $\overline{g}_x \in B_\varepsilon(\max_i g_{y_i})$. $\overline{g}_x(x) \in B_\varepsilon(f(x))$. $\overline{g}_x > f - 2\varepsilon$. This is from above. From below, similarly take $\{x_i\}$ s.t $f + 2\varepsilon > \min_i g_{x_i}$. $\exists g \in A$, $g \in B_\varepsilon(\min_i \overline{g}_{x_i})$, $\max g \in B_{3\varepsilon}(f)$

Peano's Lemma: let $F : \mathbb{R}^2 \to \mathbb{R}$ cont and consider $y : [0, T] \to \mathbb{R}$, $y(0) = a$, $y'(t) = F(t, y(t)$ on $[0, T]$. $\forall a \in \mathbb{R}$ $\exists T > 0$ s.t solution exists on the interval.

Idea: Since F cont, for $t \approx 0, y \approx a, y' \approx F(0, a)$. Control y' using Lipschitz. Create approx seq y_n where we expect limit to solve. Use compactness to show limit exists, and then show its the solution. Consider a physics application: start at a travel at speed M for time T. Still in $[0, \tau) \times B_\delta(a)$. Consider example where $f'' = -f$, $f(0) = 0$, and $f'(0) = 1$. Then we can write $f' = g$ and then we have $g' = -f$. This gives

$$\begin{bmatrix} f' \\ g' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \implies f'' = g'$$

Abbreviations

cmpt - compact
cont - continuous
conv - converge(s/nt/nce)
cplt - complete
cpt - compact
incr - increasing
LHS - left hand side
lp - limit point
nbhd - neighborhood
part - partition
unif - uniform(ly)
WLOG - without loss of generality
WTS - want to show